AN EXCEPTIONAL COLLECTION FOR KHOVANOV HOMOLOGY

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ABSTRACT. Refinements of Jones-Wenzl projectors are introduced and studied. Analogues of these new projectors are constructed within Khovanov's framework for the categorification of the Jones polynomial. Together they form exceptional collections which control the categories underlying Khovanov homology. As a consequence we obtain Postnikov decompositions of all tangle invariants.

1. INTRODUCTION

In [15], Khovanov introduced a categorification of the Jones polynomial. In subsequent papers [16, 3], this homological invariant of links was refined to a local invariant of tangles, taking values in categories $\mathrm{Kom}(n)$. This line of investigation has attracted attention from low-dimensional topologists because Khovanov's work represents a sophisticated refinement of the information encoded by the Jones polynomial.

Constructions of 3-dimensional topological quantum field theories stemming from the Jones polynomial utilize an idempotent called the Jones-Wenzl projector p_n in an essential way. The importance of the Jones-Wenzl projector can also be understood representation theoretically, it corresponds to projection onto the largest irreducible representation $V_n \subset V^{\otimes n}$ where V is the standard 2-dimensional representation of $\mathrm{U}_q\,\mathfrak{sl}(2)$.

In [6], analogues of the Jones-Wenzl projectors were defined as objects, $P_n \in \mathrm{Kom}(n)$, in the categorical setting described by Khovanov and Bar-Natan. See also [18, 9]. The intricate structure of the categorified projectors has suggested a new richness within this knot homology theory, [10, 11, 20]. Unfortunately, they alone are not sufficient to decompose the categories upon which Khovanov homology is built.

In this article we introduce and study refined Jones-Wenzl projectors, p_ϵ and $p_{n,k}$, associated to other components of the tensor product $V^{\otimes n}$. The projector P_n is used to construct chain complexes P_ϵ and $P_{n,k}$ in $\mathrm{Kom}(n)$ which categorify these refined projectors. The higher order projectors $P_{n,k}$ are shown to be unique up to homotopy and satisfy a number of useful properties.

In section 7, duality statements are developed and applied to compute mapping spaces. It is shown that the projectors P_ϵ and $P_{n,k}$ satisfy a strong form of orthogonality with respect to the Hom pairing: if $\epsilon \ntrianglelefteq \nu$ or $j < i$ then

$$\mathrm{Hom}^*(P_\epsilon, P_\nu) \simeq 0 \quad \text{or} \quad \mathrm{Hom}^*(P_{n,i}, P_{n,j}) \simeq 0.$$

There is a way to order the projectors so that the space of all maps from higher projectors to lower projectors is contractible. In other words, the refined projectors form exceptional collections.

In section 9, the resolution of identity R_n is constructed as an iterated extension of projectors P_ϵ. The complex R_n is called the resolution of identity because,

$$1_n \simeq R_n,$$

where 1_n is the identity object. For example, R_4 is pictured below.

$$\begin{tikzcd}
 & & P_{(1,-1,1,-1)} \arrow[r] \arrow[d] & P_{(1,-1,1,1)} \arrow[r] \arrow[d] \arrow[dd, bend right] & P_{(1,1,1,1)} \\
1_4 & \simeq & P_{(1,1,-1,-1)} \arrow[r] & P_{(1,1,-1,1)} \arrow[d] \arrow[ur, bend right] & \\
 & & & P_{(1,1,1,-1)} \arrow[uur, bend right] &
\end{tikzcd}$$

These projectors and the associated resolution of identity R_n are new tools for understanding Khovanov homology. For each object, $X \in \mathrm{Kom}(n)$, after tensoring the resolution of identity with X, the left-hand side becomes $X = X \otimes 1_n$ and the right-hand side becomes a homotopy theoretic decomposition of X in terms of complexes $X \otimes P_\epsilon$. In section 10, we find that factoring the resolution of identity produces a Postnikov type decomposition of X into components, $X \otimes P_{n,k}$, each of which is uniquely characterized up to homotopy. In particular, if X is the invariant associated to any tangle then X can be decomposed in this manner.

An important idea in representation theory is the decomposition of representations into direct sums of irreducible representations. In our categorical setting, a direct sum of irreducible representations is replaced by a twisted complex of idempotents. In particular, we find that there are *non-trivial* maps between, previously non-interacting, irreducible summands. In this sense, our work highlights a fundamental difference between categorified representation theory and representation theory.

2. Temperley-Lieb Category and Higher order Projectors

In this section we summarize basic information about the Temperley-Lieb category, TL, the Temperley-Lieb algebra, TL_n, and the Jones-Wenzl projectors, $p_n \in \mathrm{TL}_n$. In section 2.5, the higher order projectors $p_{n,k} \in \mathrm{TL}_n$ are defined representation theoretically. In section 2.8, refined projectors p_ϵ are introduced and related to the

higher order projectors. This allows us to characterize $p_{n,k}$ uniquely in terms of its interaction with other Temperley-Lieb elements. It is this latter definition which will lift to chain complexes in section 10. For more information about the Temperley-Lieb algebra and its connection to low-dimensional topology see [14].

2.1. **Temperley-Lieb Category.** Here we define the Temperley-Lieb category TL and establish some basic notions, such as the through-degree $\tau(a)$ of elements $a \in$ TL.

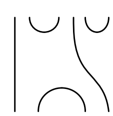
Definition 2.2. The Temperley-Lieb category TL is the category of $\mathrm{U}_q\,\mathfrak{sl}(2)$-equivariant maps from n-fold to k-fold tensor powers of the fundamental representation V.

More specifically, the objects of TL are indexed by integers n corresponding to tensor powers, $V^{\otimes n}$, of the fundamental representation and the morphisms

$$\mathrm{TL}(n,k) = \mathrm{Hom}_{\mathrm{U}_q\,\mathfrak{sl}(2)}(V^{\otimes n}, V^{\otimes k}). \tag{2.1}$$

are determined by equivariant maps. It is standard that elements of $\mathrm{TL}(n,k)$ can be represented by $\mathbb{C}(q)$-linear combinations of pictures consisting of chords from a collection of n points to a collection of k points situated on two horizontal lines in the plane. Pictures are considered equivalent if they are isotopic relative to the boundary. We also impose the relation that a disjoint circle can be removed at the cost of multiplying by $q + q^{-1}$.

A sample element of the space of morphisms from four points to six points is pictured below.

 $\in \mathrm{TL}(4,6)$

When elements are represented by pictures, the composition

$$\mathrm{TL}(n,k) \otimes \mathrm{TL}(k,l) \to \mathrm{TL}(n,l) \quad \text{where} \quad a \otimes b \mapsto ba$$

in the Temperley-Lieb category corresponds to vertical stacking.

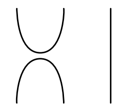
Definition 2.3. (TL_n) For each positive integer n, the usual Temperley-Lieb algebra is given by the endomorphisms,

$$\mathrm{TL}_n = \mathrm{TL}(n,n) = \mathrm{End}_{\mathrm{U}_q\,\mathfrak{sl}(2)}(V^{\otimes n}),$$

of the nth tensor power of the fundamental representation.

Elements of TL_n are generated by elementary diagrams e_i containing $n-2$ vertical chords and two horizontal chords connecting the ith and the $i+1$st positions. For

instance,

$e_1 =$  .

Definition 2.4. (through-degree) Suppose that $a \in \mathrm{TL}(n,m)$ is a Temperley-Lieb diagram then there are many ways in which a factors as a composition $a = cb$ where

$$b \otimes c \in \mathrm{TL}(n,l) \otimes \mathrm{TL}(l,m).$$

The *through-degree* $\tau(a)$ of a is equal to the minimal l achieved in such a factorization. If $a \in \mathrm{TL}(n,m)$ is a linear combination of Temperley-Lieb diagrams, $a = \sum_i f_i a_i$, then the through-degree of a is defined by

$$\tau(a) = \max_i \{\tau(a_i) \mid f_i \neq 0\}.$$

Example 1. *The through-degrees of the two illustrations above are two and one respectively.*

Through-degree can only decrease when composing elements of TL. In this manner, the category TL is filtered by through-degree.

Instead of through-degree, one might choose instead to filter elements $a \in \mathrm{TL}_n$ by the number of turnbacks, $\cap(a) = (n - \tau(a))/2$. While theorem 2.15 section 2.8 and definition 10.4 section 10 can be stated in terms of turnbacks, it is awkward to use $\cap(a)$ for general elements of the category TL.

For any element $x \in \mathrm{TL}(n,k)$ there is an element $x \sqcup 1 \in \mathrm{TL}(n+1,k+1)$ obtained by adjoining a single vertical strand to all of the diagrams appearing in the expression for x.

Given an element $x \in \mathrm{TL}(n,k)$ there is a corresponding element $\bar{x} \in \mathrm{TL}(k,n)$ obtained by flipping the diagrams representing x upside down. This operation is q-linear.

2.5. **Idempotents from irreducibles.** In this section we will explain the connection between $\mathrm{U}_q\,\mathfrak{sl}(2)$ representation theory and higher order Jones-Wenzl projectors. These higher order Jones-Wenzl projectors will be explored in section 2.8 and categorified in section 9.

The irreducible representations V_k of $\mathrm{U}_q\,\mathfrak{sl}(2)$ are indexed by integers $k \in \mathbb{Z}_{\geq 0}$. The trivial representation V_0 is 1-dimensional and the fundamental representation $V_1 \cong V$ is 2-dimensional. In general, we can use the rule

$$V_n \otimes V_1 \cong V_{n+1} \oplus V_{n-1} \tag{2.2}$$

to decompose the tensor product $V^{\otimes n}$ into a direct sum of irreducible representations:

$$V^{\otimes n} \cong V_n \oplus m_{n-2} V_{n-2} \oplus m_{n-4} V_{n-4} \oplus \cdots .$$

For each summand $m_k V_k \subset V^{\otimes n}$, there are equivariant projection and inclusion maps,

$$V^{\otimes n} \to m_k V_k \to V^{\otimes n}$$

and equation (2.1) implies the existence of a idempotent $p_{n,k} \in \mathrm{TL}_n$ in the nth Temperley-Lieb algebra because $\mathrm{TL}_n = \mathrm{Hom}_{\mathrm{U}_q\,\mathfrak{sl}(2)}(V^{\otimes n}, V^{\otimes n})$.

Definition 2.6. The idempotent $p_{n,k} \in \mathrm{TL}_n$ corresponding to the summand $m_k V_k \subset V^{\otimes n}$ is called the kth *higher order Jones-Wenzl projector.*

In the remainder of this section we will provide several descriptions of these idempotents. A different discussion can be found in [8, 19].

2.7. **Jones-Wenzl projectors.** The Jones-Wenzl Projectors $p_n \in \mathrm{TL}_n$ are the idempotent elements of the Temperley-Lieb algebra obtained from the composition

$$V^{\otimes n} \to V_n \to V^{\otimes n},$$

projecting and including onto the largest irreducible summand $V_n \subset V^{\otimes n}$.

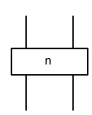
The projectors can be defined by the recurrence relation, $p_1 = 1$ and

$$p_n = p_{n-1} - \frac{[n-1]}{[n]} p_{n-1} e_{n-1} p_{n-1} \tag{2.3}$$

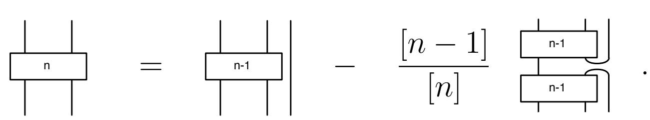
where the quantum integer $[n]$ is defined to be

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-(n-1)} + q^{-(n-3)} + \cdots + q^{n-3} + q^{n-1}.$$

If we depict p_n graphically by a box with n incoming and outgoing chords

$p_n =$ 

then formula (2.3) can be illustrated as follows



It can be shown that the Jones-Wenzl projectors are uniquely characterized by the following properties:

(1) $p_n \in \mathrm{TL}_n$.
(2) $p_n - 1$ belongs to the subalgebra generated by $\{e_1, e_2, \ldots, e_{n-1}\}$

(3) $e_i p_n = p_n e_i = 0$ for all $i = 1, \ldots, n-1$.

For more information see [22, 14, 6].

Remark. Although the coefficient ring $\mathbb{C}(q)$ is used throughout section 2. We will consistantly interpret expressions like $[n]/[n+1]$ as a power series in the ring $\mathbb{Z}[q^{-1}][[q]]$, see [6, 7] for discussion.

2.8. **Higher Order Jones-Wenzl Projectors.** Recall from section 2.5 that, for each $n > 0$, the $\mathrm{U}_q\,\mathfrak{sl}(2)$ representation $V^{\otimes n}$ decomposes as a sum of irreducible representations V_k,

$$V^{\otimes n} \cong \bigoplus_k m_k V_k.$$

Each summand $m_k V_k \subset V^{\otimes n}$ is a sum of m_k copies of the irreducible representation V_k. For each $p_{n,k}$ of definition 2.6 we would like to write

$$p_{n,k} = \sum_{\epsilon \in \mathcal{L}_{n,k}} p_\epsilon \tag{2.4}$$

where $|\mathcal{L}_{n,k}| = m_k$ and each p_ϵ projects onto a copy of V_k within $m_k V_k$ where $|\epsilon| = k$.

In order to accomplish this, we begin with a few definitions that will simplify the language used to index these refined idempotents p_ϵ (see proposition 2.13 below).

Definition 2.9. A *sequence* ϵ is an n-tuple,

$$\epsilon = (i_1, i_2, \ldots, i_n) \quad \text{where} \quad i_k \in \{-1, 1\} \text{ for } 1 \leq k \leq n.$$

The *length*, $l(\epsilon)$, of a sequence $\epsilon = (i_1, i_2, \ldots, i_n)$ is given by $l(\epsilon) = n$ and the *size*, $|\epsilon|$, of ϵ is defined to be $|\epsilon| = i_1 + \cdots + i_n$. If $\epsilon = (i_1, i_2, \ldots, i_n)$ and $\delta = (j_1, j_2, \ldots, j_n)$ are two sequences then $\epsilon \trianglerighteq \delta$ if

$$i_1 + \cdots + i_k \geq j_1 + \cdots + j_k$$

for all $k = 1, \ldots, n$.

For any sequence ϵ, if we denote by 0 the $l(\epsilon)$-tuple consisting entirely of zeros then the sequence ϵ is *admissible* if $\epsilon \trianglerighteq 0$.

We denote by $\mathcal{L}_n$ the collection of all admissible sequences of length n and $\mathcal{L}_{n,k} \subset \mathcal{L}_n$ the collection of admissible sequence of length n and size k.

$$\mathcal{L}_n = \{\epsilon : l(\epsilon) = n, \epsilon \trianglerighteq 0\} \quad \text{and} \quad \mathcal{L}_{n,k} = \{\epsilon \in \mathcal{L}_n : |e| = k\}$$

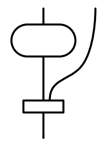
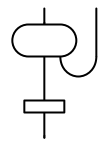
The relation $\trianglerighteq$ when applied to these sets is called the *dominance order*.

If $\epsilon = (i_1, i_2, \ldots, i_n)$ is a sequence then we will use the notation $\epsilon \cdot (+1)$ and $\epsilon \cdot (-1)$ to denote the sequence obtained from ϵ by appending 1 and -1 respectively.

$$\epsilon \cdot (+1) = (i_1, i_2, \ldots, i_n, +1) \quad \text{and} \quad \epsilon \cdot (-1) = (i_1, i_2, \ldots, i_n, -1)$$

Associated to each $\epsilon \in \mathcal{L}_n$ is a special element $q_\epsilon \in \mathrm{TL}_n$. Since these special elements are vertically symmetric, it is easiest to define the top half t_ϵ first.

Definition 2.10. (t_ϵ) If $\epsilon \in \mathcal{L}_n$ and $|\epsilon| = k$ then there is an element $t_\epsilon \in \mathrm{TL}(k, n)$ defined inductively by $t_{(1)} = 1$,

$t_{\epsilon\cdot(+1)} =$  and $t_{\epsilon\cdot(-1)} =$ 

where the box represents a Jones-Wenzl projector p_k and the marshmallow-shaped region represents the element t_ϵ.

Definition 2.11. (q_ϵ) The special element q_ϵ is equal to the top t_ϵ composed with its reverse,

$$q_\epsilon = t_\epsilon \bar{t}_\epsilon.$$

Given a sequence $\epsilon \in \mathcal{L}_{n,k}$, the recurrence

$$q_\epsilon \sqcup 1 = q_{\epsilon\cdot(+1)} + \frac{[k]}{[k+1]} q_{\epsilon\cdot(-1)} \tag{2.5}$$

can be observed by applying recurrence (2.3) of section 2.7 to the middle Jones-Wenzl projector p_k in q_ϵ.

We will use the special elements $q_\epsilon \in \mathrm{TL}_n$ to construct idempotents corresponding to the decomposition described in section 2.5. The following two propositions tell us that there are scalars $f_\epsilon \in \mathbb{C}(q)$ such that the collection $p_\epsilon = f_\epsilon q_\epsilon$ satisfy

(1) $p_\epsilon p_\nu = \delta_{\epsilon,\nu} p_\epsilon$
(2) $1_n = \sum_{\epsilon\in\mathcal{L}_n} p_\epsilon$

where $1_n \in \mathrm{TL}_n$ is the identity element. The first proposition below tells us that composing p_ϵ and p_ν when $\epsilon \neq \nu$ yields zero. The second proposition addresses the second equation and the first equation when $\epsilon = \nu$.

Proposition 2.12. *The special elements $q_\epsilon \in \mathrm{TL}_n$ defined above are mutually orthogonal.*

$$q_\epsilon q_\nu = 0 \quad \text{for} \quad \epsilon \neq \nu$$

Proof. Using the definition of q_ϵ found above we can write $q_\epsilon = a\bar{a}$ and $q_\nu = b\bar{b}$. If $\epsilon \neq \nu$ then $a = a'p_k$ and $b = b'p_l$ where $k > l$. The product $q_\nu q_\epsilon$ contains $p_k a' b' p_l$ which is equal to zero. By symmetry, $q_\epsilon q_\nu$ also vanishes. □

In the next proposition, we show that, for each $\epsilon \in \mathcal{L}_n$, there are constants $f_\epsilon \in \mathbb{C}(q)$ and idempotents $p_\epsilon = f_\epsilon q_\epsilon$ which yield the decomposition of identity $1_n \in \mathrm{TL}_n$ mentioned above.

Proposition 2.13. *For each $\epsilon \in \mathcal{L}_n$, there are idempotents $p_\epsilon \in \mathrm{TL}_n$ which satisfy,*

$$1_n = \sum_{\epsilon \in \mathcal{L}_n} p_\epsilon.$$

Moreover, $p_\epsilon = f_\epsilon q_\epsilon$ for some non-zero scalar $f_\epsilon \in \mathbb{C}(q)$.

Proof. The proof is by induction on the number of strands n. When n is 1 the only sequence is $\epsilon = (1)$; we set $f_\epsilon = 1$ so that $p_\epsilon = q_\epsilon = 1$.

Assume that there is a decomposition of 1_{n-1} and place a disjoint strand next to everything. We have

$$1_{n-1} \sqcup 1 = 1_n = \sum_{\epsilon \in \mathcal{L}_{n-1}} f_\epsilon \, q_\epsilon \sqcup 1 = \sum_k \sum_{\epsilon \in \mathcal{L}_{n-1,k}} f_\epsilon \, q_\epsilon \sqcup 1,$$

in which the elements $p_\epsilon = f_\epsilon q_\epsilon$ are idempotent. The recurrence relation (2.5) implies that

$$1_n = \sum_k \sum_{\epsilon \in \mathcal{L}_{n-1,k}} f_\epsilon \left(q_{\epsilon \cdot (+1)} + \frac{[k]}{[k+1]} q_{\epsilon \cdot (-1)} \right). \tag{2.6}$$

Setting $p_{\epsilon \cdot (+1)} = f_\epsilon q_{\epsilon \cdot (+1)}$ and $p_{\epsilon \cdot (-1)} = f_\epsilon \frac{[k]}{[k+1]} q_{\epsilon \cdot (-1)}$ yields the equation in the statement of this proposition.

To show that the p_ν are idempotent multiply $1_n = \sum_\epsilon p_\epsilon$ on the left with p_ν the previous proposition implies

$$p_\nu = \sum_\epsilon p_\nu p_\epsilon = p_\nu p_\nu.$$

□

Notice that $p_\epsilon p_\nu = 0$ if $\epsilon \neq \nu$ by proposition 2.12 because the projectors p_ϵ differ from the elements q_ϵ by scalars. The construction in the proof of proposition 2.13 above implies the equation below.

$$p_\epsilon \sqcup 1 = p_{\epsilon \cdot (+1)} + p_{\epsilon \cdot (-1)} \tag{2.7}$$

By convention $p_\epsilon = 0$ if $\epsilon \notin \mathcal{L}_n$. This equation corresponds to the decomposition rule (2.2) of section 2.5.

Each idempotent p_ϵ corresponds to projection onto one term $V_k \subset m_k V_k \subset V^{\otimes n}$. The higher order projectors $p_{n,k}$ correspond to the entire $m_k V_k \subset V^{\otimes n}$.

Definition 2.14. (higher order projectors) The kth *higher order Jones-Wenzl projector* $p_{n,k} \in \mathrm{TL}_n$ is given by the sum,

$$p_{n,k} = \sum_{\epsilon \in \mathcal{L}_{n,k}} p_\epsilon.$$

From propositions 2.12 and 2.13 it follows that the elements $p_{n,k} \in \mathrm{TL}_n$ form a complete system of mutually orthogonal idempotents. This means that $p_{n,k}p_{n,l} = \delta_{k,l}p_{n,k}$ and

$$1_n = \sum_k p_{n,k}.$$

Since $p_{n,k}$ is a sum of elements p_ϵ with $|\epsilon| = k$ and each p_ϵ necessarily factors through a Jones-Wenzl projector p_k, the projector $p_{n,k}$ factors through p_k. We record this observation below.

Observation 1. The element $p_{n,k} \in \mathrm{TL}_n$ is a linear combination of diagrams which factor as bp_ka where $a \otimes b \in \mathrm{TL}(n,k) \otimes \mathrm{TL}(k,n)$ and $p_k \in \mathrm{TL}_k$ is the kth Jones-Wenzl projector of section 2.7.

Although a definition of the higher order projectors $p_{n,k}$ was given in section 2.5, it is often useful to characterize elements intrinsically in terms of their interaction with other elements. This is the definition given below and the one which will lift to the categorical setting in section 10.

Theorem 2.15. *The higher order Jones-Wenzl projectors*

$$p_{n,k} \in \mathrm{TL}_n$$

of definition 2.14 are characterized uniquely by the following properties

(1) *The through-degree $\tau(p_{n,k})$ of $p_{n,k}$ is equal to k.*
(2) *The projector $p_{n,k}$ vanishes when the number of turnbacks is sufficiently high. For any $l \in \mathbb{Z}_+$ and $a \in \mathrm{TL}(n,l)$ if $\tau(a) < k$ then*

$$ap_{n,k} = 0 \quad \text{and} \quad p_{n,k}\bar{a} = 0.$$

(3) *The projector $p_{n,k}$ fixes elements of through-degree k up to lower through-degree terms. For any $l \in \mathbb{Z}_+$ and $a \in \mathrm{TL}(n,l)$ if $\tau(a) = k$ then*

$$ap_{n,k} = a + b$$

where $\tau(b) < k$.

In essence, these properties state that the projectors $p_{n,k}$ control and respect the filtration of TL by through-degree τ, see also the discussion following definition 2.4.

Proof. We begin by proving that the elements $p_{n,k}$ defined above satisfy properties (1)-(3). Using observation 1 above, we can write $p_{n,k}$ as a sum of the form,

$$p_{n,k} = \sum_d dp_k\bar{d}.$$

The first property follows from $\tau(p_k) = k$. Now pick some $l \in \mathbb{Z}_+$ and $a \in \mathrm{TL}(n,l)$.

For the second property, if we assume that $\tau(a) < k$ then

$$ap_{n,k} = \sum_d adp_k\bar{d} = 0$$

since $\tau(ad) \leq \tau(a) < k$ and p_k kills diagrams of through-degree less than k. For the same reason $p_{n,k}\bar{a} = 0$.

For the third property, if we assume that $\tau(a) = k$ then

$$a = a1_n = \sum_l ap_{n,l} = \sum_{l \leq k} ap_{n,l},$$

so that rearranging terms gives $ap_{n,k} = a - \sum_{l<k} ap_{n,l}$.

Suppose that $e \in \mathrm{TL}_n$ satisfies properties (1)-(3) above, we will show that $e = p_{n,k}$.

If $l < k$ then property (2) for e implies that $ep_{n,l} = 0$. If $l > k$ then property (2) for $p_{n,l}$ implies that $ep_{n,l} = 0$. Therefore,

$$e = e1_n = \sum_l ep_{n,l} = ep_{n,k}.$$

Property (3) implies that $ep_{n,k} = p_{n,k} + b$ where $\tau(b) < k$. Therefore,

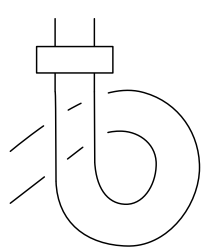
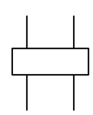
$$e = ep_{n,k} = ep_{n,k}^2 = (p_{n,k} + b)p_{n,k} = p_{n,k}^2 = p_{n,k},$$

using proposition 2.13.

□

We continue our discussion of the higher order projectors with a series of observations.

There is a standard extension of TL_n to tangles, see [14]. If $T_n \in \mathrm{TL}_n$ is the element represented by the full twist then one can show that $T_np_n = q^{2n}p_n$.


$= q^{2n}$

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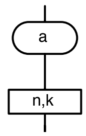
From observation 1, it follows that

$$T_np_{n,k} = q^{2k}p_{n,k}.$$

Observation 2. For each element $a \in \mathrm{TL}(n,m)$, the equation $ap_{n,k} = p_{m,k}a$ holds.



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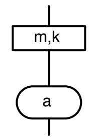
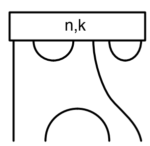


Proof. Recall that $1_m = \sum_l p_{m,l}$ and $1_n = \sum_l p_{n,l}$. Simplifying the resulting expressions for $1_m a p_{n,k}$ and $p_{m,k} a 1_n$ gives $ap_{n,k} = p_{m,k} a p_{n,k} = p_{m,k} a$. $\square$

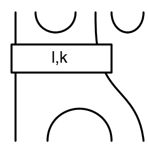
In particular, if $D \in \mathrm{TL}_n$ and the through-degree $\tau(D) = l$ so that $D = ba$ where $a \otimes b \in \mathrm{TL}(n,l) \otimes \mathrm{TL}(l,n)$ then

$$p_{n,k} D = p_{n,k} ba = b p_{l,k} a.$$

This means we can slide a $p_{n,k}$ past some turnbacks onto a fewer number of strands as long as we change it to a $p_{l,k}$. In pictures,



$$=$$



.

Note that the highest higher order Jones-Wenzl projector $p_{n,n} \in \mathrm{TL}_n$ is the Jones-Wenzl projector p_n of [22].

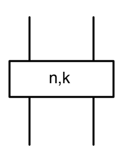
Observation 3. Using the recurrence 2.3 in section 2.7, we can write

$$p_n = 1_n - \sum_{D \in \mathrm{TL}_n} f_D D$$

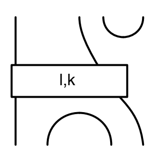
where $f_D \in \mathbb{C}(q)$ and the sum is over diagrams D not equal to the identity diagram. Multiplying both sides by $p_{n,k}$ and using orthogonality yields,

$$p_{n,k} = \sum_{\tau(D)<n} f_D p_{n,k} D = \sum_{\tau(D)<n} f_D b_D p_{\tau(D),k} a_D,$$

where the second equality follows because the through-degree $\tau(D) < n$ of each D in the sum and we apply observation 2. This is an inductive formula for $p_{n,k}$ in terms of p_n and $p_{l,k}$ for $l < n$. Graphically,



$$= \sum_{\tau(D)<n} f_D$$



.

Iterating this procedure yields a formula for $p_{n,k}$ purely in terms of p_k.

3. Categorification of the Temperley-Lieb Category

In this section we recall Dror Bar-Natan's graphical formulation [3] of the Khovanov categorification [15, 16]. We follow the same conventions as [6].

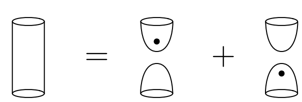
There is a pre-additive category Pre-Cob(n) whose objects are isotopy classes of formally q-graded Temperley-Lieb diagrams with $2n$ boundary points. The morphisms are given by $\mathbb{Z}[\alpha]$ linear combinations of isotopy classes of orientable cobordisms, decorated with dots, and bounded in $D^2 \times [0,1]$ between two disks containing such diagrams. The *degree* of a cobordism $C : q^i A \to q^j B$ is given by

$$\deg(C) = \deg_\chi(C) + \deg_q(C)$$

where the topological degree $\deg_\chi(C) = \chi(C) - n$ is given by the Euler characteristic of C and the q-degree $\deg_q(C) = j - i$ is given by the relative difference in q-gradings. The maps C used throughout the paper will satisfy $\deg(C) = 0$. The formal q-grading will be chosen to cancel the topological grading.

When working with chain complexes, every object will also contain a homological grading and every map will have an associated homological degree. Homological degree, or t-degree, is not part of the definition $\deg(C)$. We may refer to degree as internal degree in order to differentiate between degree and homological degree.

We impose the relations below to obtain a new category Cob(n) as a quotient of the category Mat(Pre-Cob(n)) formed by allowing direct sums of objects and maps between them.



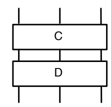
The dot is determined by the relation that two times a dot is equal to a handle. The cylinder or neck cutting relation implies that closed surfaces Σ_g of genus $g > 3$ evaluate to 0. In what follows we will let α be a free variable and absorb it into our base ring ($\Sigma_3 = 8\alpha$). One can think of α as a deformation parameter, see [3].

The categories Cob(n) fit together in much the same way as the Temperley-Lieb algebras TL$_n$. There is an inclusion $- \sqcup 1_{m-n} : \mathrm{Cob}(n) \to \mathrm{Cob}(m)$ whenever $n \leq m$ obtained by unioning each diagram with $m - n$ disjoint vertical line segments on the right to each object and $m - n$ disjoint disks to each morphism. If $m = n$ then the empty set is used instead of either intervals or disks.

There is a category $\mathrm{Cob}(m,n)$ with objects corresponding to diagrams in $\mathrm{TL}(m,n)$, so that $\mathrm{Cob}(n) = \mathrm{Cob}(n,n)$. There is a composition

$$\otimes : \mathrm{Cob}(n,k) \times \mathrm{Cob}(k,l) \to \mathrm{Cob}(n,l) \quad \text{where} \quad A \times B \mapsto B \otimes A$$

obtained by gluing all diagrams and morphisms along the k boundary points and k boundary intervals respectively. Pictorially,

$C \otimes D =$ 

.

This composition makes the collection of categories $\mathrm{Cob}(n,k)$ into a 2-category Cob. The relationship between Cob and the Temperley-Lieb category TL can be described using the Grothendieck group functor K_0.

Theorem 3.1. *The 2-category* Cob *categorifies the Temperley-Lieb category* TL*. There are isomorphisms,*

$$\mathrm{TL}(n,k) \cong \mathrm{K}_0(\mathrm{Cob}(n,k)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}(q)$$

which commute with composition.

These isomorphisms commute with compositions in the appropriate sense. For more detail see [3, 16, 7].

Definition 3.2. Let $\mathrm{Kom}(n,m) = \mathrm{Kom}(\mathrm{Cob}(n,m))$ be the category of chain complexes of objects in $\mathrm{Cob}(n,m)$.

Unless otherwise stated chain complexes are bounded only in negative homological degree. All chain complexes produced in what follows will have differentials with components having internal degree zero. Restricting to the subcategory of chain complexes with degree zero differentials yields a well-behaved Grothendieck group functor $\mathrm{K}_0(\mathrm{Kom}(n,k))$, see [6].

The usual category of chain complexes can be enriched to form a differential graded category.

Definition 3.3. There is a differential graded category, $\mathrm{Kom}^*(n,m)$, which has the same objects as $\mathrm{Kom}(n,m)$ but morphisms given by allowing maps of all homological degrees.

If $f \in \mathrm{Hom}^m(A,B)$ is a map of homological degree m then $d(f) = [d,f] \in \mathrm{Hom}^{m+1}(A,B)$. Each collection $\mathrm{Hom}^*(A,B)$, of morphisms from A to B in $\mathrm{Kom}^*(n,m)$, is a chain complex and the differential d is a derivation with respect to composition of maps.

4. The Category of Twisted Complexes

In this section we recall the definition of the category $\mathrm{Tw}\,\mathcal{A}$ of twisted complexes over a differential graded category $\mathcal{A}$, see [2, 13]. The reader may assume that $\mathcal{A} = \mathrm{Kom}^*(n,m)$, see definition 3.3 section 3.

Our main construction in section 9 will occur in the category of twisted complexes. Informally, the definitions in this section codify situations in which the objects of study are chain complexes M with a decreasing filtration

$$M = F^0M \supseteq F^1M \supseteq F^2M \supseteq \dots$$

and a splitting $F^{i+1}(M) = G_{i+1} \oplus F^i(M)$ as graded objects. Maps are required to respect this filtration.

The definitions presented here are variations on standard ones which allow one to work with categories of twisted complexes which are unbounded and which are indexed by countable sets (such as $\mathbb{Z}_+$). This is accomplished by requiring that maps are lower triangular, see definition 4.3 below.

Definition 4.1. (twisted complex) A *twisted complex* over $\mathcal{A}$ is a collection

$$\{(E_i), q_{ij} : E_i \to E_j\} \quad \text{where} \quad i \in \mathbb{Z}_+$$

consisting of objects $E_i \in \mathcal{A}$ and maps q_{ij} of degree $i-j+1$ which satisfy $q_{ij} = 0$ for $i \geq j$ and the equation

$$(-1)^j d_{\mathcal{A}}(q_{ij}) + \sum_k q_{kj} \circ q_{ik} = 0. \tag{4.1}$$

Twisted complexes which satisfy the condition that $q_{ij} = 0$ when $i \geq j$ are called *one-sided*. In this document, we require all twisted complexes to be one-sided.

It will be convenient later to use ordered sets besides $\mathbb{Z}_+$ to index components of twisted complexes. In particular, the set of sequences $\mathcal{L}_n$ together with the dominance order described in definition 2.9 section 2.8 will be used throughout section 9. In general, it will be clear from context when this is done.

Definition 4.2. ($\mathrm{Tw}\,\mathcal{A}$) The one-sided twisted complexes form a differential graded category. If $A = \{(A_i), a_{ij}\}$ and $B = \{(B_i), b_{ij}\}$ then degree k maps are those that intertwine the diagrams formed by A and B,

$$\mathrm{Hom}^k_{\mathrm{Tw}\,\mathcal{A}}(A,B) = \prod_{i \leq j} \mathrm{Hom}^{k+i-j}_{\mathrm{Kom}^*(n,m)}(A_i, B_j).$$

In other words, morphisms $f : A \to B$ are collections $\{f_{ij}\}$ of maps having the appropriate degree which satisfy $f_{ij} = 0$ unless $i \leq j$. Composition of morphisms is defined in terms of components by the equation,

$$(f \circ g)_{ij} = \sum_{i \leq k \leq j} f_{jk} \circ g_{ik}.$$

If $f \in \mathrm{Hom}^*(A,B)$ is given by $\{f_{ij}\}$ then the equation

$$(df)_{ij} = (-1)^j d_{\mathcal{A}}(f_{ij}) + \sum_k b_{kj} \circ f_{ik} - (-1)^{|f|} f_{kj} \circ q_{ik}$$

determines a differential which makes $\mathrm{Tw}\,\mathcal{A}$ into a differential graded category.

The categories $\mathrm{Tw}\,\mathrm{Kom}^*(n,m)$ are examples of pre-triangulated categories. Pre-triangulated categories can be seen as an alternative to triangulated categories because every such category $\mathcal{A}$ yields a triangulated category $H^0(\mathcal{A})$, see [2].

If $\{E_i\} \subset \mathrm{Kom}(\mathcal{C})$ is a collection of non-negatively graded chain complexes then as graded objects, $\prod_{i\geq 0} t^i E_i \cong \bigoplus_{i\geq 0} t^i E_i$ since the direct product is finite in each degree. This allows us to flatten each twisted complex $A = \{(A_i), a_{ij}\}$ to a chain complex $\mathrm{Tot}(A)$ by summing together the individual components A_i of A.

Definition 4.3. If $\mathcal{A} = \mathrm{Kom}(\mathcal{C})$ is the category of non-negatively graded chain complexes over an additive category $\mathcal{C}$ then there is a dg functor

$$\mathrm{Tot} : \mathrm{Tw}\,\mathcal{A} \to \mathcal{A}$$

from twisted complexes to complexes defined on objects $\{(E_i), q_{ij}\} \in \mathrm{Tw}\,\mathcal{A}$ by

$$\mathrm{Tot}(\{(E_i), q_{ij}\}) = \left\{ \bigoplus_{i\geq 0} t^i E_i \ , \ d \right\} \qquad \text{where} \qquad d = \begin{pmatrix} d_{E_0} & & & \\ q_{01} & -d_{E_1} & & \\ q_{02} & q_{12} & d_{E_2} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and on morphisms $f = \{f_{ij}\}$ by

$$\mathrm{Tot}(f) = \begin{pmatrix} f_{00} & & & \\ f_{01} & f_{11} & & \\ f_{02} & f_{12} & f_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The condition that $d^2_{\mathrm{Tot}(A)} = 0$ is implied by equation (4.1) above.

The functor Tot defined above preserves homotopy equivalences. In particular, for all $X, Y \in \mathrm{Tw}\,\mathcal{A}$,

$$X \simeq Y \Rightarrow \mathrm{Tot}(X) \simeq \mathrm{Tot}(Y).$$

Remark. If $\mathcal{C}$ is an additive category which contains countable direct products and $\mathcal{A}$ is the differential graded category of possibly unbounded chain complexes over $\mathcal{C}$ then we can define a dg functor

$$\mathrm{Tot}^{\Pi} : \mathrm{Tw}\,\mathcal{A} \to \mathcal{A}$$

using the formulas above with $\oplus$ replaced by Π.

Definition 4.4. (convolution) If a chain complex A is the total complex of some twisted complex $\{(E_i), q_{ij}\}$ then we say A is a *convolution* of the E_i.

Example 2. *In lemma 5.4 of section 5 the twisted complex T pictured below is considered.*

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rr, bend left, "\rho"] \arrow[rrr, bend right, "\eta"'] & tB \arrow[r, "\beta"] \arrow[rr, bend left, "\sigma"] & t^2C \arrow[r, "\gamma"] & t^3D
\end{tikzcd}$$

Each object A, B, C and D is a chain complex. The convolution $\mathrm{Tot}(T)$ *is the chain complex*

$$A \oplus tB \oplus t^2C \oplus t^3D$$

with differential

$$d_{\mathrm{Tot}(T)} = \begin{pmatrix} d_A & & & \\ \alpha & d_B & & \\ \rho & \beta & d_C & \\ \eta & \sigma & \gamma & d_D \end{pmatrix}.$$

This twisted complex is one-sided with respect to the order of the letters appearing in the alphabet.

The notion of hull defined below formalizes the idea of the subcategory of all chain complexes built out of iterated extensions of elements of some fixed set of chain complexes.

Definition 4.5. (hull) If $\mathcal{E} = \{A_1, \ldots, A_r\} \subset \mathrm{Kom}(\mathcal{C})$ is a collection of chain complexes then the *hull* $\langle \mathcal{E} \rangle \subset \mathrm{Kom}(\mathcal{C})$ is the smallest strictly full additive subcategory containing each A_i and closed under convolution.

In particular, if $\{(E_i), q_{ij}\} \in \mathrm{Tw}\,\mathrm{Kom}(\mathcal{C})$ satisfies $E_i \in \langle \mathcal{E} \rangle$ for all $i \in \mathbb{Z}_+$ then $\mathrm{Tot}(\{(E_i), q_{ij}\}) \in \langle \mathcal{E} \rangle$.

If $\mathcal{E} \subset \mathcal{A}$ is a collection of objects then the Grothendieck group of the hull of $\mathcal{E}$ is the span of $\mathrm{K}_0(\mathcal{E})$ in the Grothendieck group of $\mathcal{A}$,

$$\mathrm{K}_0(\langle \mathcal{E} \rangle) = \langle \mathrm{K}_0(\mathcal{E}) \rangle.$$

Definition 4.6. (cone) Suppose that $A = \{(A_i), a_{ij}\}$ and $B = \{(B_i), b_{ij}\}$ are twisted complexes and $f = \{f_{ij}\} : A \to B$ is a degree zero cycle then the *cone* of f is the twisted complex given by

$$\mathrm{Cone}(f) = \left\{ (A_i \oplus B_{i-1}) \; , \; \begin{pmatrix} a_{ij} & 0 \\ f_{i,j-1} & -b_{i-1,j-1} \end{pmatrix} \right\}.$$

The condition that $\mathrm{Cone}(f)$ is a twisted complex is equivalent to the requirement that f is a degree zero cycle in the definition above.

If A, B are twisted complexes over a category of chain complexes and $f : A \to B$ a degree zero cycle then

$$\mathrm{Tot}(\mathrm{Cone}(f)) = \mathrm{Cone}(\mathrm{Tot}(f)).$$

Definition 4.7. (truncation of a twisted complex) Suppose that $Y = \{(Y_i), y_{ij}\}$ is a twisted complex indexed by $\mathbb{Z}_+$. For any $a, b \in \mathbb{Z}_+$ with $a \leq b$ the $[a,b]-$*truncation* of Y is given by

$$Y_{[a,b]} = \{(T_i), t_{ij}\}$$

where $T_i = Y_i$, $t_{ij} = y_{ij}$ when $i, j \in [a,b]$ and $T_i = 0$, $t_{ij} = 0$ when $i, j \notin [a,b]$.

The lemma below says that a twisted complex is determined by its truncations and each truncation is an iterated mapping cone. For simplicity of notation, we restrict to $\mathbb{Z}_+$ indexed twisted complexes over the categories $\mathrm{Kom}^*(n)$.

Lemma 4.8. *If $\{(E_i), q_{ij}\}$ is a twisted complex over $\mathrm{Kom}^*(n)$ then for each integer $s \geq 0$,*

$$\mathrm{Tot}(Y_{[0,s]}) = \mathrm{Cone}(\mathrm{Tot}(Y_{[0,s-1]}) \xrightarrow{\delta} t^{s-1}E_s),$$

where

$$\delta = \begin{pmatrix} q_{0,s} & q_{1,s} & \dots & q_{s-1,s} \end{pmatrix}$$

is a chain map of degree zero. Conversely, if we have chain complexes C_s and maps $\delta_s : C_s \to t^s E_{s+1}$ such that $C_{s+1} = \mathrm{Cone}(\delta_s)$ then there is a unique twisted complex Y such that $C_s = \mathrm{Tot}(Y_{[0,s]})$.

The next lemma is a useful tool for showing that certain filtered chain complexes are contractible.

Recall that an object E in a dg category $\mathcal{A}$ is *contractible* if Id_E is a boundary in $\mathrm{End}_{\mathcal{A}}(E)$.

Lemma 4.9. *If $\{E_i\} \subset \mathcal{A}$ is a collection of contractible objects then any twisted complex $\{(E_i), q_{ij}\}$ is contractible.*

One subtlety to keep in mind is that the corresponding result for chain complexes only holds in situations where convolution Tot is defined, e.g. over a category of non-negatively graded chain complexes *or* a category of possibly unbounded chain complexes over an additive category containing countable direct products.

The following theorem says that the dg subcategory of $\mathcal{A}$ determined by the hull of $\mathcal{E}$ is controlled by the dg algebra of Hom-spaces between objects in $\mathcal{E}$.

Theorem 4.10. *([2]) If $\mathcal{E}$ is a collection of objects in a pre-triangulated category $\mathcal{A}$ then the category of differential graded modules over the algebra*

$$E = \bigoplus_{i,j} \mathrm{Hom}^*(E_i, E_j)$$

is equivalent to the category of $\langle \mathcal{E} \rangle$.

One is primarily interested in applying this theorem in cases when Hom-spaces between objects in the collection $\mathcal{E}$ satisfy nice properties. It is common to ask for a condition such as the one defined below.

Definition 4.11. A family of objects $\mathcal{E} = \{E_1, E_2, \ldots, E_n\}$ is an *exceptional collection* if

$$\mathrm{Hom}^*(E_i, E_j) \simeq 0 \quad \text{for} \quad i > j.$$

If $\mathcal{E}$ is an exceptional collection and $\langle \mathcal{E} \rangle = \mathcal{A}$ then the differential algebra

$$A = \bigoplus_{i \leq j} \mathrm{Hom}^*(E_i, E_j)$$

controls the category $\mathcal{A}$, in the sense that A-mod $\simeq \mathcal{A}$. Theorem 10.9 section 10 applies this theorem with $\mathcal{P}^n = \{P_{n,n}, P_{n,n-2}, P_{n,n-4}, \ldots\}$. The higher order projectors form an exceptional collection and so determine a nice differential graded algebra E^n controlling a complete system of idempotents on Khovanov homology.

5. Homotopy Lemmas

In this section we introduce a number of lemmas that will be used repeatedly throughout sections 8 and 9.

We will use the following proposition in theorem 9.1 section 9 to construct the chain complexes P_ϵ.

Proposition 5.1. *(obstruction theory) For any pair of chain maps $\alpha : A \to B$ and $\beta : B \to C$ there is a chain map $\gamma : \mathrm{Cone}(\alpha) \to tC$ of the form $\gamma = \begin{pmatrix} -h & \beta \end{pmatrix}$ where $\beta \circ \alpha = d_C \circ h$ if and only if $\beta \circ \alpha \simeq 0$.*

Moreover, if $\mathrm{Hom}^*(A, C) \simeq 0$ then $\beta \circ \alpha \simeq 0$ and the map γ is unique up to homotopy.

Proof. The associated homotopy category Ho(Kom) is triangulated. There is an exact triangle,

$$A \to B \to \mathrm{Cone}(\alpha).$$

Applying the functor $\mathrm{Hom}(-, C)$ yields a long exact sequence,

$$\cdots \to \mathrm{Hom}^i(\mathrm{Cone}(\alpha), C) \to \mathrm{Hom}^i(B, C) \to \mathrm{Hom}^i(A, C) \to \cdots.$$

If $\beta \circ \alpha \simeq 0$ then $\alpha^*(\beta) = 0$ and exactness implies the existence of γ. One can check that γ is given by the map above between chain complexes. Uniqueness of γ is implied by exactness on the other side. $\square$

The uniqueness of the lifts γ in the proposition above will guarantee that there is exactly one choice at each stage in the construction of the projectors P_ϵ, see theorem 9.1 and corollary 9.2 section 9.

It will be useful to add a contractible chain complex to a chain complex using the Cone construction.

Lemma 5.2. *(substitution) Let A, B, C, D be chain complexes and $f : B \to D$ a chain map. Then we have:*

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rr, bend left, "\sigma"] & B \arrow[r, "\beta"] & C & \simeq & A \arrow[r, "\gamma"] \arrow[rr, bend left, "\sigma"] \arrow[dr, bend right, "f \circ \alpha"'] & \mathrm{Cone}(f) \arrow[r, "\delta"] & C \\
 & & & & & D \arrow[u, hook, "\zeta"] &
\end{tikzcd}
$$

where $\gamma = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$, $\delta = \begin{pmatrix} \beta & 0 \end{pmatrix}$ *and* $\zeta = \begin{pmatrix} 0 \\ \mathrm{Id} \end{pmatrix}$.

The following corollary is an inductive version of the previous lemma.

Corollary 5.3. *(inductive substitution) Suppose that $A = \{(A_i), q_{ij}\}$ is a one-sided twisted complex and there are maps $f_i : A_i \to C_i$ then $A \simeq \{(C_i \hookrightarrow B_i), p_{ij}\}$ where $B_i = \mathrm{Cone}(f_i)$.*

Note that we may have $C_i = 0$ for some i in the above corollary.

The next lemma allows us to remove arrows between objects in a twisted complex when the Hom-spaces between these objects is contractible (as in theorem 7.9 section 7).

Lemma 5.4. *(Combing Lemma) If $\beta \in \mathrm{Hom}^*(B, C)$ is a boundary then*

$$
\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[rr, bend left, "\rho"] \arrow[rrr, bend right, "\eta"'] & B \arrow[r, "\beta"] \arrow[rr, bend left, "\sigma"] & C \arrow[r, "\gamma"] & D & \cong & A \arrow[r, "\alpha"] \arrow[rr, bend left, "\rho + h\alpha"] \arrow[rrr, bend right, "\eta"'] & B \arrow[rr, bend left, "\sigma - \gamma h"] & C \arrow[r, "\gamma"] & D
\end{tikzcd}
$$

In words, we may remove β from the right-hand side. However, in doing so, we perturb the differential by arrows which factor through α or γ. Note that, if $\mathrm{Hom}^*(B, C) \simeq 0$ then every $f \in \mathrm{Hom}^*(B, C)$ is a boundary.

Proof. Since β is a boundary there exists a homotopy $h : B \to C$ such that $\beta = dh - hd$. This allows us to define maps $\varphi = \varphi^1, \varphi^{-1}$ where

$$\varphi^{\pm 1} = \begin{pmatrix} \mathrm{Id}_A & & & \\ & \mathrm{Id}_B & & \\ & \pm h & \mathrm{Id}_C & \\ & & & \mathrm{Id}_D \end{pmatrix}.$$

Notice that $\varphi\varphi^{-1} = \mathrm{Id}$ and $\varphi^{-1}\varphi = \mathrm{Id}$. If d is the differential on the left-hand side then $\varphi d_{\mathrm{Tot}(T)}\varphi^{-1}$ is the differential on the right-hand side.

□

6. Universal Projectors

In this section we recall the categorified Jones-Wenzl projector P_n. Notation for degree shifts is introduced. We conclude with a lemma regarding a specific form of the projector P_n.

In [6] a special idempotent element $P_n \in \mathrm{Kom}(n)$ was defined which categorifies the Jones-Wenzl projector.

Theorem 6.1. *([6]) There exists a chain complex $P_n \in \mathrm{Kom}(n)$ called the* universal projector *which satisfies*

(1) *P_n is positively graded with differential having internal degree zero.*
(2) *The identity diagram appears only in homological degree zero and only once.*
(3) *The chain complex P_n is contractible under turnbacks.*

These three properties characterize P_n uniquely up to homotopy.

See also [9, 18]. See [4] and [21] for related ideas.

We now remind the reader how degree shifts are denoted. Each chain complex can be shifted in both q-degree and a t-degree or homological degree.

If A is a chain complex then tA will denote the chain complex shifted in homological degree by 1,

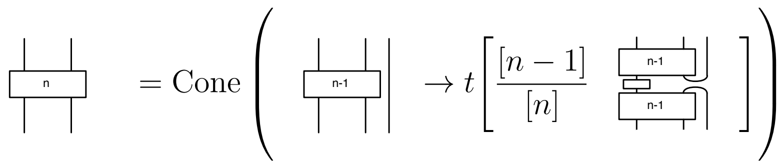
$$(tA)_i = A_{i-1} \quad \text{and} \quad d_{tA} = -d_A$$

We will use qA to denote the chain complex satisfying $\deg_q(qB) = \deg_q(B)+1$ where $B \in \text{Pre-Cob}(n)$ corresponds to a summand of A, see section 3 for a discussion of q-degree.

If $C \in \mathrm{Kom}(n,m)$ is a chain complex and $f(q) \in \mathbb{Z}[q^{-1}][[q]]$ is a power series then we will write $[f(q)C]$ for any iterated cone of chain complexes $A_0, A_1, \ldots$ in which $A_i = C$ for all i such that, in the Grothendieck group $\mathrm{K}_0(\mathrm{Kom}(n,m))$, we have

$$[f(q)C] = f(q) \cdot [D].$$

Lemma 6.2. *If $P_n \in \mathrm{Kom}(n)$ is a projector then there is a twisted complex*



which is also a projector.

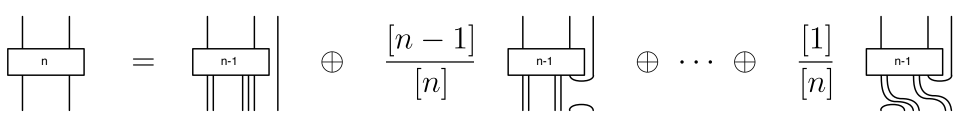
Proof. The proof follows from tensoring the Frenkel-Khovanov complex of [6] for P_n with $(P_{n-2} \sqcup 1_2) \otimes e_{n-1} \otimes (P_{n-1} \sqcup 1)$ then contracting portions of the subcomplex consisting of projectors containing turnbacks. □

Definition 6.3. (brackets for epsilons) If C is a chain complex of the form Q_ϵ where $\mathrm{K}_0(Q_\epsilon) = q_\epsilon$ then we will consistently omit a product of terms of the form $[k]/[k+1]$ from the bracket notation. Usually,

$$[Q_\epsilon] = [f_\epsilon Q_\epsilon]$$

where f_ϵ is defined in the proof of proposition 2.13.

If $f(q) \in \mathbb{Z}[q^{-1}][[q]]$ and C is a chain complex then one may also denote a sum of graded objects by $f(q)C$. This allows us to express some results more concisely. For example, it follows from the construction of the universal projector in [6] that P_n is a convolution in the form of the Frenkel-Khovanov sequence FK_n



as graded objects.

7. Computing Spaces of Maps

In this section we recall a duality for Hom-spaces inside of the category $\mathrm{Kom}(n, m)$. For each sequence $\epsilon \in \mathcal{L}_n$, chain level analogues $Q_\epsilon \in \mathrm{Kom}(n)$ of the elements $q_\epsilon \in \mathrm{TL}_n$ found in section 2.8 are introduced. In theorem 7.9 the duality statement is used to prove that Hom-spaces between convolutions of Q_ϵ and Q_ν respect the dominance order.

7.1. **Duality.** Denote by $\mathrm{Kom}(n)^b$ the subcategory of $\mathrm{Kom}(n)$ consisting of chain complexes which are bounded on both sides in homological degree.

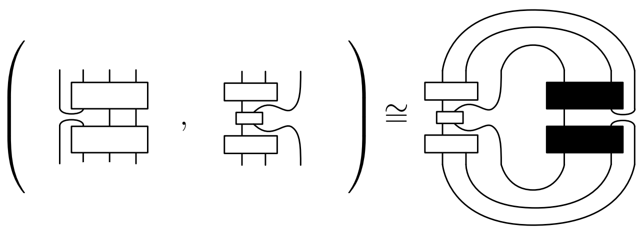
Definition 7.2. (reflection) If $C \in \mathrm{Kom}(n)^b$ then a new complex $C^\vee$ is obtained using the automorphism which reflects all of the diagrams in the chain complex about the x-axis and reverses both the quantum and the homological gradings.

For instance, when defining the invariants of tangles which live in $\mathrm{Kom}(n)^b$ (see [3, 16]) the chain complex associated to a negative crossing can be obtained from the chain complex associated to a positive crossing by applying this functor.

One can show that $(C^\vee)^\vee \cong C$ and that $-^\vee : \mathrm{Kom}(n)^b \to \mathrm{Kom}(n)^b$ preserves homotopy. Our primary interest in $-^\vee$ stems from its behavior with respect to the pairing

$$\mathrm{Kom}(n)^b \times \mathrm{Kom}(n)^b \to \mathrm{Kom}(0)^b \quad \text{where} \quad (X,Y) \mapsto \mathrm{Hom}^*(X,Y).$$

More specifically, the computation of Hom-spaces in $\mathrm{Kom}(n)^b$ can be simplified using the planar algebra trick,

Hom^* 

which allows us to describe the Hom-pairing in terms of a gluing of chain complexes and duals of chain complexes.

In the illustration above, the boxes represent choices of chain complexes in $\mathrm{Kom}(n)^b$ for various n, so that on the left-hand side of this equation is the chain complex of maps between the two objects in the differential graded category $\mathrm{Kom}^*(n)^b$ (see section 3). On the right-hand side of this equation is the chain complex in $\mathrm{Kom}(0)^b$ formed by dualizing the first of the two objects and then connecting its free end points to those of the second object. This is identified with the chain complex of abelian groups on the left after applying the functor $\mathrm{Hom}(\emptyset, -)$.

Theorem 7.3. *For all $A, B \in \mathrm{Kom}(n)^b$, the Markov trace of the chain complex $B \otimes A^\vee \in \mathrm{Kom}(n)^b$ computes the extended* Hom*-space from A to B.*

$$\mathrm{Hom}^*(A,B) \cong \mathrm{Tr}(B \otimes A^\vee)$$

Proof. If $f : D \to D'$ is a map between diagrams D and D' in $\mathrm{Cob}(n)$ then there is a canonical way to construct an element of $D'^\vee \otimes D$. This commutes with maps and so respects the differential. $\square$

This duality has been explored in [5] theorem 1.3, in [19] this duality was denoted by $-^{\Diamond}$ and in [4] by $\mathbb{D}(-)$. Moreover, in any rigid monoidal category there is an isomorphism

$$\mathrm{Hom}(1, X^{\vee} \otimes Y) \to \mathrm{Hom}(X, Y)$$

see [1]. In our setting we can identify the right-hand side with the chain complex determined by the Markov trace.

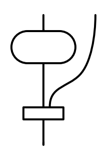
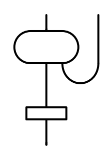
Since the category $\mathrm{Kom}(n)^b$ is closed under $-^{\vee}$ it has a bit of extra symmetry lacking in the category $\mathrm{Kom}(n)$. One way to remedy this is to allow one term in the Hom-pairing to be a chain complex in $\mathrm{Kom}(n)$ and require the other term to be a chain complex in $\mathrm{Kom}(n)^b$.

$$\mathrm{Kom}(n)^b \times \mathrm{Kom}(n) \to \mathrm{Kom}(0) \quad \text{or} \quad \mathrm{Kom}(n) \times \mathrm{Kom}(n)^b \to \mathrm{Kom}(0)$$

When this is the case Theorem 7.3 above continues to hold. This is all that is necessary for the proof of theorem 7.9 below. For an alternative viewpoint, see [12].

7.4. **Categorical Quasi-Idempotents.** In this section we associate to each sequence $\epsilon \in \mathcal{L}_n$ a special chain complex $Q_\epsilon \in \mathrm{Kom}(n)$. This construction is directly analogous to the definition of q_ϵ in section 2.8. Since these special complexes are vertically symmetric, it is easiest to define the top half T_ϵ first.

Definition 7.5. (T_ϵ) If $\epsilon \in \mathcal{L}_n$ and $|\epsilon| = k$ then there is an element $T_\epsilon \in \mathrm{Kom}(k, n)$ defined inductively by $T_{(1)} = 1$,

$T_{\epsilon\cdot(+1)} =$  and $T_{\epsilon\cdot(-1)} =$ 

where the box represents a universal projector P_k (theorem 6.1 section 6) and the marshmallow-shaped region represents the element T_ϵ.

Definition 7.6. (Q_ϵ) The special element $Q_\epsilon \in \mathrm{Kom}(n)$ is equal to the top T_ϵ composed with its reverse,

$$Q_\epsilon = T_\epsilon \otimes \bar{T}_\epsilon.$$

In other words, replacing p_k with P_k in the definition of q_ϵ gives us Q_ϵ. The graded Euler characteristic of Q_ϵ is the element of TL_n obtained from q_ϵ after identifying its coefficients with elements of $\mathbb{Z}[[q]][q^{-1}]$. These chain complexes will be used extensively in sections 8 and 9.

7.7. **Hulls of Q_ϵ are Perpendicular.** Before stating the main theorem in this section, we must introduce a lemma which will be used in its proof.

Lemma 7.8. *If $N \in \mathrm{Kom}(n)$ then the collection of complexes annihilated by N,*

$$\mathrm{Ann}(N) = \{M \in \mathrm{Kom}(n) : M \otimes N \simeq 0\}$$

is closed under convolution.

Proof. If $\{(E_i), q_{ij}\}$ is a twisted complex with $E_i \in \mathrm{Ann}(N)$ then

$$\mathrm{Tot}(\{(E_i), q_{ij}\}) \otimes N \cong \mathrm{Tot}(\{(E_i \otimes N), q_{ij} \otimes \mathrm{Id}_N\}) \simeq 0$$

by lemma 4.9. □

In particular, if $Q_\epsilon \otimes N \simeq 0$ then $\langle Q_\epsilon \rangle \subset \mathrm{Ann}(N)$ and $[Q_\epsilon] \in \langle Q_\epsilon \rangle \Rightarrow [Q_\epsilon] \otimes N \simeq 0$.

The following theorem tells us that any convolution of Q_ϵ is perpendicular to any convolution of Q_ν with respect to the Hom-pairing when $\epsilon \ntrianglelefteq \nu$. This is used in theorem 9.1 in conjunction with proposition 5.1 to inductively construct the projectors P_ϵ.

Theorem 7.9. *If $\epsilon, \nu \in \mathcal{L}_n$ are sequences and $\epsilon \ntrianglelefteq \nu$ then*

$$\mathrm{Hom}^*([Q_\epsilon], [Q_\nu]) \simeq 0$$

for any $[Q_\epsilon] \in \langle Q_\epsilon \rangle$ and $[Q_\nu] \in \langle Q_\nu \rangle$.

Proof. Suppose that $n \in \mathbb{Z}_+$ and $N = [Q_\nu]^n$ is the nth chain group of the chain complex $[Q_\nu]$. The condition $\epsilon \ntrianglelefteq \nu$ implies that there is an i such that

$$\epsilon_1 + \cdots + \epsilon_i > \nu_1 + \cdots + \nu_i,$$

let $k = \epsilon_1 + \cdots + \epsilon_i$ and $l = \nu_1 + \cdots + \nu_i$. By definition of Q_ν, every summand a of $N = [Q_\nu]^n$ can be written as

$$a = c \otimes (b \sqcup 1_{n-i}) \quad \text{for some} \quad b \in \mathrm{TL}(i, l)$$

and b satisfies $Q_\epsilon \otimes (b \sqcup 1_{n-i})^\vee \simeq 0$ (for the same reasons as proposition 2.12). Hence, $Q_\epsilon \otimes N^\vee \simeq 0$. Lemma 7.8 and theorem 7.3 imply that

$$\mathrm{Hom}^*([Q_\epsilon], [Q_\nu]^n) = \mathrm{Hom}^*([Q_\epsilon], N) = \mathrm{Hom}^*([Q_\epsilon] \otimes N^\vee, 1_n) \simeq 0.$$

Finally, we observe that $\mathrm{Hom}^*([Q_\epsilon], [Q_\nu]) = \mathrm{Tot}^\Pi(E)$ (see remark section 4) where

$$E = \mathrm{Hom}^*([Q_\epsilon], [Q_\nu]^0) \to \mathrm{Hom}^*([Q_\epsilon], [Q_\nu]^1) \to \ldots.$$

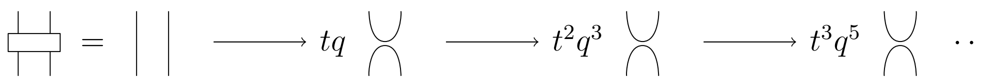
So $\mathrm{Hom}^*([Q_\epsilon], [Q_\nu])$ is a convolution of contractible chain complexes and lemma 4.9 implies the theorem. □

8. Explicit Constructions of Resolutions of Identity

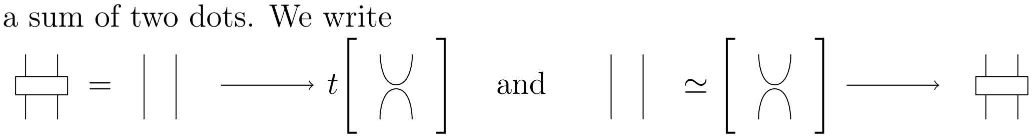
In this section the higher order projectors are constructed for $n = 2, 3,$ and 4 strands. While the construction for $n > 4$ (section 9) is fairly involved, all of the important features can be seen concretely when $n = 4$.

The subscripts used in this section correspond to the indexing convention introduced in definition 2.9 of section 2.8.

8.1. **Two strands: $P_{(1,-1)}$ and $P_{(1,1)}$.** The second projector P_2 can be represented by chain complex of the form



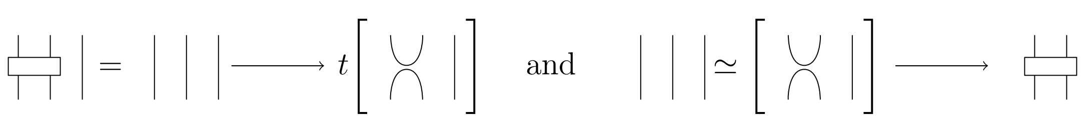
where the first map is a saddle and the last two maps alternate between a difference and a sum of two dots. We write



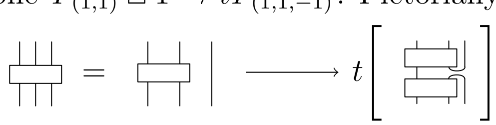
where the map defining the first cone is the saddle appearing in the definition of P_2 and the map in the second cone is the inclusion of the tail into P_2. Let us write $tP_{(1,-1)}$ for the subcomplex of P_2 consisting of terms in homological degree greater than zero and set $P_{(1,1)} = P_2$. There is a map $i : P_{(1,-1)} \to P_{(1,1)}$, with $\deg_t(i) = 1$, which satisfies $\mathrm{Cone}(i) \simeq 1_2$ where 1_2 is the identity diagram illustrated above.

The chain complex $P_{(1,-1)}$ is idempotent and the map i gives the resolution of identity.

8.2. **Three Strands: $P_{(1,-1,1)}$, $P_{(1,1,-1)}$ and $P_{(1,1,1)}$.** The identity object 1_3 on three strands is given by the union of the identity object on two strands together with an extra strand, $1_3 = 1_2 \sqcup 1_1$. Applying $- \sqcup 1$ to the resolution of identity in the previous section we obtain

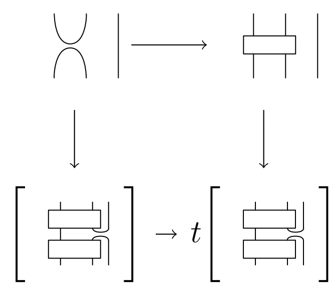


Lemma 6.2 implies that the third universal projector $P_3 = P_{(1,1,1)}$ can be chosen to be equal to the cone $P_{(1,1)} \sqcup 1 \to tP_{(1,1,-1)}$. Pictorially,

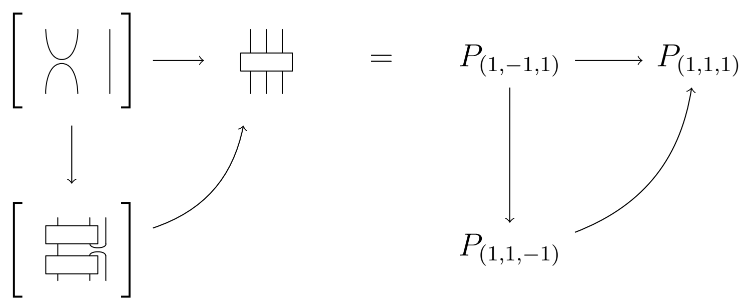


(8.1)

Consider the contractible chain complex $\mathrm{Cone}(-\mathrm{Id}) = P_{(1,1,-1)} \to tP_{(1,1,-1)}$. Using the second equation above and from gluing on the contractible chain complex it follows that 1_3 is homotopic to

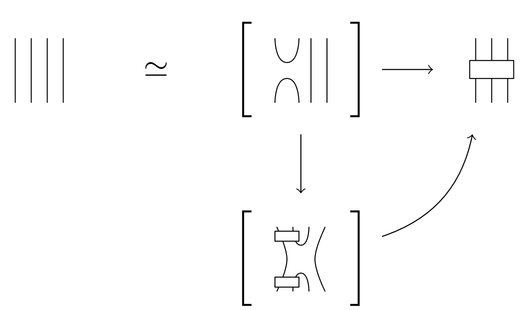


by lemma 5.2. Using the triangle (8.1) above and reassociating allows us to write this complex in terms of the projectors $P_{(1,-1,1)} = P_{(1,-1)} \sqcup 1$, $P_{(1,1,-1)}$ and $P_{(1,1,1)}$. The identity object 1_3 is homotopy equivalent to R_3.

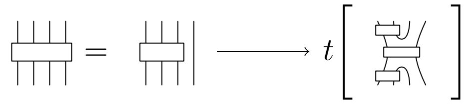


The maps above are compositions of inclusions of tails and differentials from chain complexes of projectors.

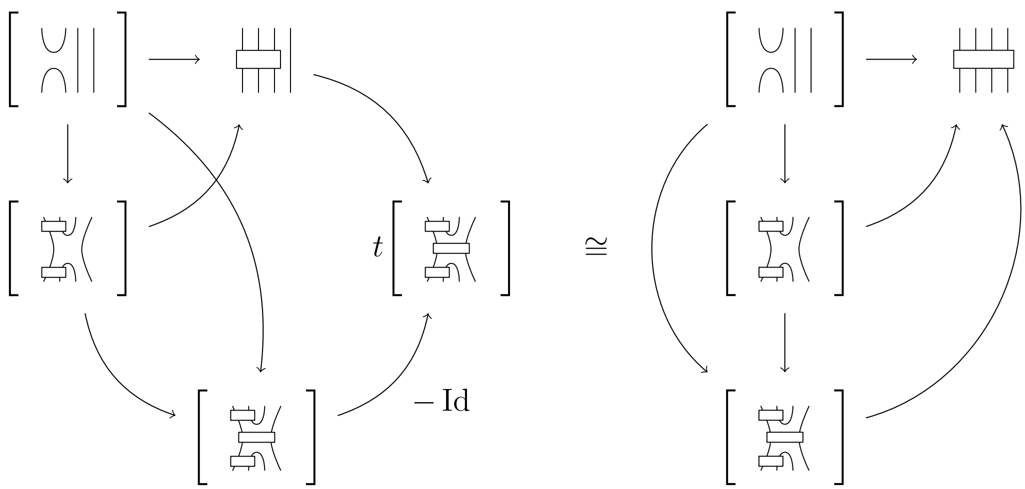
8.3. **Four strands.** In the previous section we obtained a resolution of the identity on three strands. The identity object 1_4 on four strands is given by the union of the identity object on three strands together with an extra strand, $1_4 = 1_3 \sqcup 1_1$. Applying $- \sqcup 1$ to the resolution of identity in the previous section we obtain the diagram pictured below.



Now lemma 6.2 implies that the fourth universal projector $P_4 = P_{(1,1,1,1)}$ can be chosen to be equal to the cone $P_3 \sqcup 1 \to tP_{(1,1,1,-1)}$.

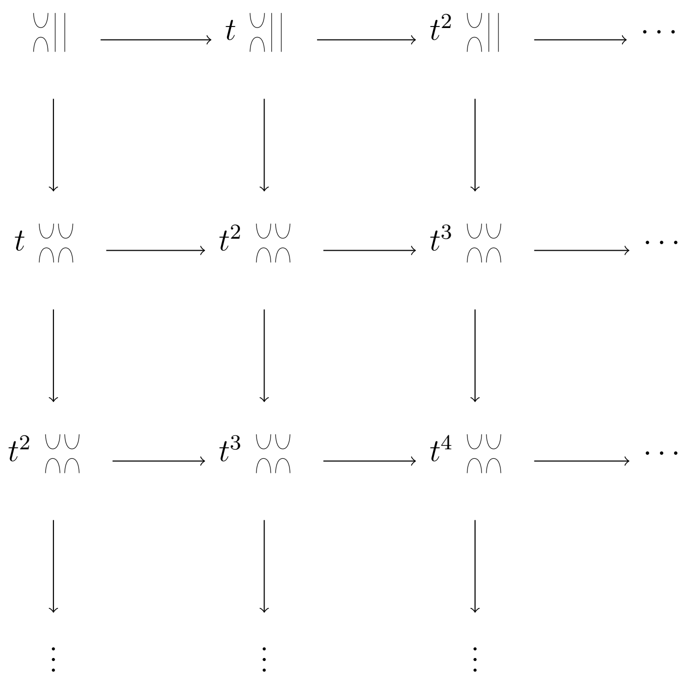


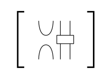
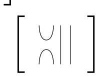
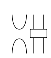
Using lemma 5.2 we can add the the contractible chain complex $\mathrm{Cone}(-\mathrm{Id}) = P_{(1,1,1,-1)} \to tP_{(1,1,1,-1)}$ to the decomposition above to obtain a homotopy equivalent complex on the left-hand side below.

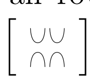


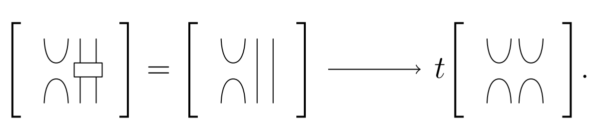
Reassociating allows us to replace $P_{(1,1,1)} \sqcup 1$ in the resolution of identity and yields the isomorphic complex containing the projector P_4 on the right-hand side above. Unfortunately, we aren't done because our resolution of identity still consists of terms which do not factor through universal projectors. In order to replace the two offending terms, $P_{(1,1,-1)} \sqcup 1$ and $P_{(1,-1,1)} \sqcup 1$, a bit of work remains. The process by which we replace $P_{(1,1,-1)} \sqcup 1$ will illustrate the general strategy.

We can construct the following chain complex,



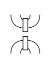
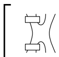
in more concise bracket notation this is $P_{(1,-1,1,1)} =$ . Notice the top row of this bicomplex is $P_{(1,-1,1)} \sqcup 1$; in bracket notation, this is . The columns of the bicomplex $P_{(1,1,-1,1)}$ are given by  .

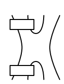
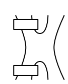
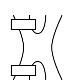
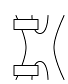
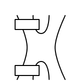
We define $P_{(1,-1,1,-1)}$ to be the tail of the bicomplex $P_{(1,1,-1,1)}$: the subcomplex consisting of all rows beyond the first (shifted down by 1). In bracket notation, $P_{(1,-1,1,-1)} =$ . The vertical differential of the bicomplex $P_{(1,1,-1,1)}$ determines a map $\delta : P_{(1,-1,1)} \sqcup 1 \to P_{(1,-1,1,-1)}$ such that $P_{(1,-1,1,1)} = \mathrm{Cone}(\delta)$. In pictures,



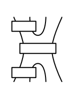
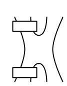
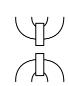
Using lemma 5.2 we can add the contractible chain complex $\mathrm{Cone}(-\mathrm{Id}) = P_{(1,-1,1,-1)} \to tP_{(1,-1,1,-1)}$ to our decomposition above and reassociate. The identity object 1_4 is homotopy equivalent to the left complex pictured below. The complex on the right is obtained by reassociating.

$$\begin{tikzcd}
P_{(1,-1,1,-1)} \arrow[d, shift left, "-\mathrm{Id}"'] & P_{(1,-1,1)} \sqcup 1 \arrow[r] \arrow[dl, bend right] \arrow[d] \arrow[dd, bend right=60] & P_{(1,1,1,1)} & & P_{(1,-1,1,-1)} \arrow[r] & P_{(1,-1,1,1)} \arrow[r] \arrow[d] \arrow[dd, bend right=60] & P_{(1,1,1,1)} \\
tP_{(1,-1,1,-1)} & P_{(1,1,-1)} \sqcup 1 \arrow[d] \arrow[ur, bend right] & & \cong & & P_{(1,1,-1)} \sqcup 1 \arrow[d] \arrow[ur, bend right] & \\
& P_{(1,1,1,-1)} \arrow[uur, bend right=60] & & & & P_{(1,1,1,-1)} \arrow[uur, bend right=60] &
\end{tikzcd}$$

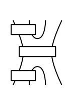
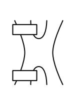
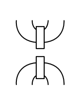
We still have to replace the subcomplex $P_{(1,1,-1)} \sqcup 1$ with a complex that factors through P_2. In order to accomplish this task we construct a chain complex $P_{(1,1,-1,-1)} =$ [] and a chain map $\gamma : P_{(1,1,-1)} \sqcup 1 \to P_{(1,1,-1,-1)}$ so that $P_{(1,1,-1,1)} =$ $\mathrm{Cone}(\gamma) =$ []. The complex $P_{(1,1,-1)} \sqcup 1 =$ [] can be written as an iterated cone

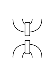
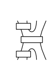
[] $=$ (((( $\to t$ ) $\to t^2$ ) $\to t^3$ ) $\to \cdots$).

We can also write

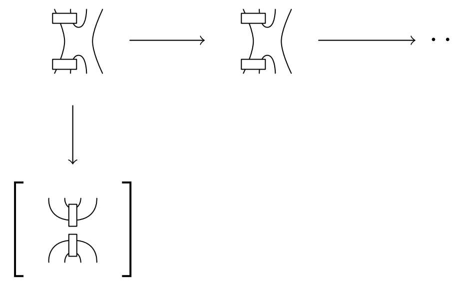
 $=$  $\longrightarrow t$ [].

Using this map we now construct a new triangle of the form

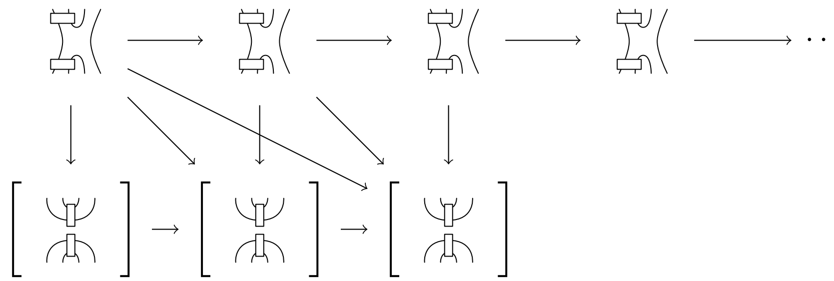
[] $=$ [] $\longrightarrow t$ [[]].

This process is also carried out in section 9 theorem 9.3. We use double brackets above to emphasize that the term on the left is a convolution of convolutions and also to distinguish it from the complex [] which is the tail of  .

The first step is to form the cone on the first term of $P_{(1,1,-1)} \sqcup 1$.

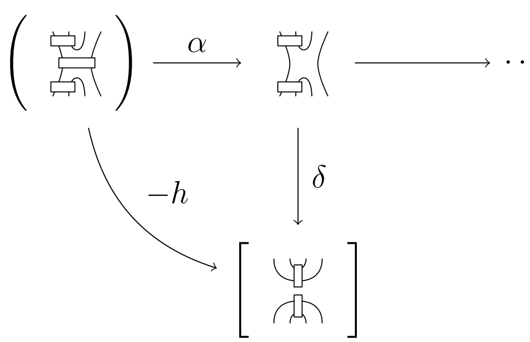


Reassociating shows that the first term in this complex agrees with the desired complex. Now assume by induction that we can form a chain complex in which the first N terms of $P_{(1,1,-1)} \sqcup 1$ have been written in this way.

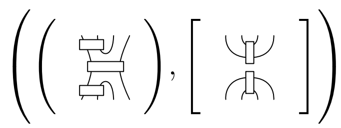


We draw the diagonal arrows to emphasize that the maps in this contruction necessarily propagate in a non-trivial way.

After grouping the first N terms of the top and bottom rows within parenthesis we consider taking the cone on the $N+1$st term



After taking shifts into account, the composition $\delta \circ \alpha$ is a chain map of degree 0 and

$\delta \circ \alpha \in \mathrm{Hom}^*$  $\simeq 0.$

The Hom-complex is contractible by theorem 7.9 section 7. Proposition 5.1 allows us to produce a chain complex with $N+1$ terms of the desired form. This process is stable, adding the $N+1$st map does not change any maps which appear earlier, because in proposition 5.1, the map γ is an extension of the map β. Since there are countably many terms we can use this process to produce the chain complex, $P_{(1,1,-1,1)}$, that we want.

By construction the top row is $P_{(1,1,-1)} \sqcup 1$ and so we define the bottom row to be $tP_{(1,1,-1,-1)}$. The non-horizontal components of the differential yield a chain map

$$\eta : P_{(1,1,-1)} \sqcup 1 \to P_{(1,1,-1,-1)}$$

such that $P_{(1,1,-1,1)} = \mathrm{Cone}(\eta)$. Our program is resumed by replacing the $P_{(1,1,-1)} \sqcup 1$ term above. By introducing the contractible term $\mathrm{Cone}(-\mathrm{Id}_{P_{(1,1,-1,-1)}}) = P_{(1,1,-1,-1)} \to tP_{(1,1,-1,-1)}$ to our last complex, we see that 1_4 is homotopy equivalent to the diagram pictured below.

$$\begin{tikzcd}
 & P_{(1,-1,1,-1)} \arrow[r] & P_{(1,-1,1,1)} \arrow[r] \arrow[dll] \arrow[d] \arrow[dd, bend right] & P_{(1,1,1,1)} \\
P_{(1,1,-1,-1)} \arrow[dd, "-\mathrm{Id}"'] & & P_{(1,1,-1)} \sqcup 1 \arrow[ur, bend right] \arrow[dll, "\eta"'] \arrow[d] & \\
tP_{(1,1,-1,-1)} & & P_{(1,1,1,-1)} \arrow[uur, bend right] &
\end{tikzcd}$$

We conclude by reassociating and using the combing lemma 5.4 of section 5 to exchange the bad arrow $P_{(1,-1,1,1)} \to P_{(1,1,-1,-1)}$ with an arrow $P_{(1,-1,1,-1)} \to P_{(1,1,-1,-1)}$ that respects the dominance order $\trianglelefteq$ on $\mathcal{L}_4$ (definition 2.9 section 2.8). The object 1_4 is homotopic to the complexes pictured below.

$$\begin{tikzcd}
P_{(1,-1,1,-1)} \arrow[r] & P_{(1,-1,1,1)} \arrow[r] \arrow[dl, bend right] \arrow[d] \arrow[dd, bend right] & P_{(1,1,1,1)} & & P_{(1,-1,1,-1)} \arrow[r] \arrow[d] & P_{(1,-1,1,1)} \arrow[r] \arrow[d] \arrow[dd, bend right] & P_{(1,1,1,1)} \\
P_{(1,1,-1,-1)} \arrow[r] & P_{(1,1,-1,1)} \arrow[d] \arrow[ur, bend right] & & \cong & P_{(1,1,-1,-1)} \arrow[r] & P_{(1,1,-1,1)} \arrow[d] \arrow[ur, bend right] & \\
& P_{(1,1,1,-1)} \arrow[uur, bend right] & & & & P_{(1,1,1,-1)} \arrow[uur, bend right] &
\end{tikzcd}$$

The end result is a resolution of identity on four strands in which all of the terms factor through universal projectors of the form P_{4-2k} for $k = 0, 1, 2$ and all maps between terms respect the dominance order.

In order to accomplish this task we needed two basic manuevers. The first was gluing a contractible chain complex onto our resolution without changing the homotopy type using lemma 5.2 in section 5. The second was the construction of chain complexes suitable for substitution, proposition 5.1 in section 4 and theorem 7.9 section 7 tell us that this is always possible.

In order to construct the P_ϵ, a general version of the argument given above is carried out in section 9. The reader may refer to this section for illustrations.

9. General Construction of the Resolution of Identity

In this section we categorify the equation

$$1_n = \sum_{\epsilon \in \mathcal{L}_n} p_\epsilon \tag{9.1}$$

of proposition 2.13 section 2.8. This is accomplished by constructing chain complexes $P_\epsilon \in \mathrm{Kom}(n)$ for each sequence $\epsilon \in \mathcal{L}_n$ which satisfy idempotence and orthogonality properties,

$$P_\epsilon \otimes P_\nu \simeq \delta_{\epsilon\nu} P_\epsilon.$$

For each $n > 0$, using the projectors P_ϵ, we construct a complex R_n, the *resolution of identity*, which satisfies,

$$1_n \simeq R_n.$$

In particular, $\mathrm{K}_0(R_n)$ can be identified with the right-hand side of equation (9.1) above.

In section 2.8, in order to define the elements p_ϵ, we began by defining $q_\epsilon \in \mathrm{TL}_n$ and we set $p_\epsilon = f_\epsilon q_\epsilon$. The definition of q_ϵ can be lifted directly to give chain complexes $Q_\epsilon \in \mathrm{Kom}(n)$. This was done in section 7.4.

In theorem 7.9 section 7, we showed that $\epsilon \ntrianglelefteq \nu$ implies

$$\mathrm{Hom}^*([Q_\epsilon], [Q_\nu]) \simeq 0, \tag{9.2}$$

for any two convolutions $[Q_\epsilon] \in \langle Q_\epsilon \rangle$ and $[Q_\nu] \in \langle Q_\nu \rangle$.

Theorem 9.1 below exploits this fact in order to build triangles relating convolutions in $\langle Q_{\epsilon\cdot(+1)} \rangle$, $\langle Q_{\epsilon\cdot(-1)} \rangle$ and $\langle Q_\epsilon \sqcup 1 \rangle$. An immediate consequence is corollary 9.2, which constructs chain complexes

$$P_\epsilon \in \langle Q_\epsilon \rangle,$$

that categorify the idempotents $p_\epsilon \in \mathrm{TL}_n$.

Theorem 9.1. *For each sequence $\epsilon \in \mathcal{L}_n$ and convolution $[Q_\epsilon] \in \langle Q_\epsilon \rangle$ there exists a convolution $[Q_{\epsilon\cdot(-1)}] \in \langle Q_{\epsilon\cdot(-1)} \rangle$ and a chain map*

$$\delta : [Q_\epsilon] \sqcup 1 \to [Q_{\epsilon\cdot(-1)}]$$

of homological and internal degree zero such that

$$\mathrm{Cone}(\delta) \in \langle Q_{\epsilon\cdot(+1)} \rangle.$$

In what follows, we define the convolution $[Q_{\epsilon\cdot(+1)}]$ to be $\mathrm{Cone}(\delta)$ because $\mathrm{Cone}(\delta)$ is canonically determined by the convolution $[Q_\epsilon]$ and the choice $[Q_{\epsilon\cdot(-1)}]$.

The proof below is a generalization of the obstruction theoretic argument used to construct the map $\eta : P_{(1,1,-1)} \sqcup 1 \to tP_{(1,1,-1,-1)}$ in section 8.3.

Proof. Let $S \subset \langle Q_\epsilon \rangle$ denote the collection of chain complexes for which the theorem is true. In order to prove the theorem we show that $Q_\epsilon \in S$ and that S is closed under convolution. These two together imply that $\langle Q_\epsilon \rangle \subset S$.

By definition, there exists $A_\epsilon \in \mathrm{Kom}^*(n)$ such that $Q_\epsilon = A_\epsilon \otimes P_k \otimes \overline{A}_\epsilon$ where $|\epsilon| = k$ and there is a triangle $P_{k+1} = P_k \sqcup 1 \xrightarrow{\delta'} tT$ where T denotes the tail of the projector P_{k+1} (lemma 6.2). Setting $[Q_{\epsilon\cdot(-1)}] = (A_\epsilon \sqcup 1) \otimes T \otimes (\overline{A}_\epsilon \sqcup 1)$ and $\delta = \mathrm{Id} \otimes \delta' \otimes \mathrm{Id}$ shows that $Q_\epsilon \in S$.

The remainder of the proof shows that S is closed under convolutions. Suppose that $[Q_\epsilon] \in \langle Q_\epsilon \rangle$ is a convolution,

$$[Q_\epsilon] = \mathrm{Tot}(E) \quad \text{where} \quad E = \{(E_i), q_{ij}\}$$

and $E_i \in \langle Q_\epsilon \rangle$ are chain complexes for which the theorem holds. By assumption there exist complexes $T_i \in \langle Q_{\epsilon\cdot(-1)} \rangle$ and maps

$$\delta_i : E_i \sqcup 1 \to T_i \quad \text{such that} \quad \mathrm{Cone}(\delta_i) \in \langle Q_{\epsilon\cdot(+1)} \rangle.$$

We wish to define a chain complex $[Q_{\epsilon\cdot(+1)}] \in \langle Q_{\epsilon\cdot(+1)} \rangle$ which, as a graded object, is a sum of the complexes appearing in the diagram below.

$$\begin{tikzcd}
E_0 \sqcup 1 \arrow[r] \arrow[d, "\delta_0"] \arrow[dr] \arrow[drr] & tE_1 \sqcup 1 \arrow[r] \arrow[d, "\delta_1"] \arrow[dr] & t^2E_2 \sqcup 1 \arrow[r] \arrow[d, "\delta_2"] & \cdots \\
tT_0 \arrow[r] & t^2T_1 \arrow[r] & t^3T_2 \arrow[r] & \cdots
\end{tikzcd}$$

The convolution $[Q_{\epsilon\cdot(+1)}]$ will be defined as a direct limit of truncations $[Q_{\epsilon\cdot(+1)}]_{[0,r]}$, see definition 4.7. We proceed by induction on r.

If $r = 0$ then set $[Q_{\epsilon\cdot(+1)}]_{[0,0]} = \mathrm{Cone}(\delta_0)$. Assume that $[Q_{\epsilon\cdot(+1)}]_{[0,r]}$ has constructed and let $[Q_\epsilon \sqcup 1]_{[0,r]}$ denote the corresponding truncation of $[Q_\epsilon] \sqcup 1$, which appears as the top row of $[Q_{\epsilon\cdot(+1)}]_{[0,r]}$. The differential on $[Q_\epsilon] \sqcup 1$ gives a chain map $\alpha : [Q_\epsilon \sqcup 1]_{[0,r]} \to t^r E_{r+1} \sqcup 1$. Let z be the composition of the maps in the diagram below.

$$[Q_{\epsilon\cdot(+1)}] \xrightarrow{\quad\pi\quad} [Q_\epsilon \sqcup 1]_{[0,r]} \xrightarrow{\quad\alpha\quad} t^r E_{r+1} \xrightarrow{(-1)^r\delta_{r+1},} t^r T_{r+1}$$

where π is the projection of $[Q_{\epsilon\cdot(+1)}]_{[0,r]}$ onto its top row. The map z belongs to a contractible Hom-space,

$$z \in \mathrm{Hom}^*([Q_{\epsilon\cdot(+1)}], t^r T_{r+1}) \simeq 0,$$

by theorem 7.9. Therefore, $z = d(h)$ is a boundary and proposition 5.1 allows us to produce a chain complex with $r + 1$ terms of the desired form.

$$\begin{tikzcd}
{[Q_{\epsilon\cdot(+1)}]_{[0,r]}} \arrow[r, "\alpha\circ\pi"] \arrow[dr, bend right, "h"'] & t^{r+1}E_{i+1} \sqcup 1 \arrow[d, "(-1)^{r+1}\delta_{r+1}"] \\
& t^{r+2}T_{r+1}
\end{tikzcd}$$

Observe that the differentials only point south east. The new column in this complex is $t^{r+1}\,\mathrm{Cone}(\delta_{r+1})$. Since π is the identity on the top row and zero on the bottom row, the top row of this complex is the corresponding truncation of $[Q_\epsilon] \sqcup 1$. This construction is stable for the same reasons as in section 8.3.

Defining $[Q_{\epsilon\cdot(+1)}]$ to be the limit of the resulting directed system completes the proof. $\square$

Corollary 9.2. *For each sequence $\epsilon \in \mathcal{L}_n$, there exist a chain complex $P_\epsilon \in \langle Q_\epsilon \rangle$ in the hull of Q_ϵ and maps*

$$\delta_\epsilon : P_\epsilon \sqcup 1 \to t P_{\epsilon\cdot(-1)}$$

such that

$$P_{\epsilon\cdot(1)} = \mathrm{Cone}(\delta_\epsilon).$$

The following observation follows from the argument given above.

Observation 4. For each $n > 0$ and each sequence $\epsilon \in \mathcal{L}_n$ the triangle

$$P_\epsilon \sqcup 1 = P_{\epsilon\cdot(-1)} \to P_{\epsilon\cdot(+1)}$$

descends to equation (2.7) of section 2.8 in the Grothendieck group $\mathrm{K}_0(\mathrm{Kom}(n))$.

The following theorem is a generalization of the resolution of identity found in section 8.3.

Theorem 9.3. *For each $n > 0$, there is a twisted complex $R_n = \{(P_\epsilon), d_{\epsilon\nu}\}_{\epsilon\in\mathcal{L}_n} \in \mathrm{Tw}\,\mathrm{Kom}(n)$ such that*

$$1_n \simeq R_n.$$

Proof. When $n = 1$ we set $R_1 = 1$. By induction there is a twisted complex $R_{n-1} = \{(P_\epsilon), d_{\epsilon\nu}\}_{\epsilon\in\mathcal{L}_{n-1}}$ such that

$$1_{n-1} \simeq R_{n-1}.$$

Placing a disjoint strand next to everything yields

$$1_n = R_{n-1} \sqcup 1 = \{(P_\epsilon \sqcup 1), d_{\epsilon\nu} \sqcup 1\}.$$

Corollary 5.3 and Corollary 9.2 imply that we can replace each $P_\epsilon \sqcup 1$ with

$$P_{\epsilon\cdot(-1)} \to P_{\epsilon\cdot(+1)}$$

obtaining an equivalence

$$1_n \simeq \left\{ P_{\epsilon\cdot(-1)} \xrightarrow{f_\epsilon} P_{\epsilon\cdot(+1)}, \begin{pmatrix} d_{P_{\epsilon\cdot(-1)}} & 0 \\ f_\epsilon & d_{P_{\epsilon\cdot(+1)}} \end{pmatrix} \right\} \cong \{P_{\epsilon\cdot(\pm 1)}, d'_{\epsilon\nu}\}$$

The right-hand side is a twisted complex indexed by elements of $\mathcal{L}_n$. There may be maps which do not respect the order $\trianglelefteq$ on $\mathcal{L}_n$. However, equation (9.2) above implies that when $\epsilon \ntrianglelefteq \nu$

$$\mathrm{Hom}^*(P_\epsilon, P_\nu) \simeq 0. \tag{9.3}$$

Applications of the combing lemma 5.4 of section 5 allow us to exchange maps in $\{P_{\epsilon\cdot(\pm 1)}, d'_{\epsilon\nu}\}$ which do not respect the dominance order for those that do. The resulting twisted complex is the resolution of identity R_n. ☐

Definition 9.4. The resolution of identity R_n is defined to be $\mathrm{Tot}(R_n)$ when referring to a chain complex in $\mathrm{Kom}(n)$.

In the Grothendieck group $\mathrm{K}_0(\mathrm{Kom}(n))$, the resolution of identity becomes the equation

$$1_n = \sum_{\epsilon \in \mathcal{L}_n} p_\epsilon$$

from section 2.8. From the discussion of section 2.5 we see that R_n categorifies the decomposition of $V_1^{\otimes n}$ into irreducible representations.

In the decategorified setting representations decompose into direct sums of irreducible representations. After categorification we have learned that this decomposition is maintained up to homotopy, but the irreducible components now have non-trivial maps between them.

10. Categorified Higher Order Projectors

In this section we will define the universal higher order projectors and articulate the sense in which the resolutions of identity produced in sections 8 and 9 yield categorifications of the idempotents defined in section 2.5. While the axioms of definition 10.4 given below are sufficient to characterize the projectors $P_{n,k}$ uniquely up to homotopy we will see that the $P_{n,k}$ also satisfy a number of other useful properties analogous to those enumerated in section 2.8. We begin by introducing a few definitions similar to those of section 2.1.

Recall from section 3 that there is a composition

$$\otimes : \mathrm{Cob}(n,k) \times \mathrm{Cob}(k,l) \to \mathrm{Cob}(n,l)$$

Just as elements $a \in \mathrm{TL}(n,m)$ have a notion of through-degree (definition 2.4 section 2.1), chain complexes $A \in \mathrm{Kom}(n,m)$ have a corresponding notion of through-degree.

Definition 10.1. (through-degree) Suppose that $A \in \mathrm{Cob}(n,m)$ is a Temperley-Lieb diagram then A factors as a composition $A = C \otimes B$ where

$$B \times C \in \mathrm{Cob}(n,l) \times \mathrm{Cob}(l,m).$$

The *through-degree* $\tau(A)$ of A is equal to the minimal l in such a factorization. If $A \in \mathrm{Kom}(n,m)$ is chain complex of Temperley-Lieb diagrams $\{a_i\}$ then $\tau(a) = \max_i \tau(a_i)$.

We now define subcategories $\mathrm{Kom}^k(n)$ of $\mathrm{Kom}(n)$. Elements of $\mathrm{Kom}^k(n)$ will be convolutions of complexes which factor through the universal projector P_k. If $A \in \mathrm{Kom}^k(n)$ then $\tau(A) = k$.

Definition 10.2. (Kom^k) A chain complex $C \in \mathrm{Kom}(n,m)$ *factors through* P_k if there exists $A \in \mathrm{Kom}(n,k)$ and $B \in \mathrm{Kom}(k,m)$ such that

$$C \cong A \otimes P_k \otimes B.$$

Let $\mathrm{Kom}^k(n,m) \subset \mathrm{Kom}(n,m)$ be the full subcategory of convolutions of chain complexes which factor through P_k. We have analogous notions of subcategories $\mathrm{Kom}^{*,k}(n,m)$ and $\mathrm{Tw}\,\mathrm{Kom}^k(n,m)$ in $\mathrm{Kom}^*(n,m)$ and $\mathrm{Tw}\,\mathrm{Kom}(n,m)$ respectively. See section 4 for definitions of these categories.

The next lemma tells us that chain complexes which factor through various universal projectors P_k compose in a predictable manner.

Lemma 10.3. *For each $A \in \mathrm{Kom}^k(n,m)$ and $B \in \mathrm{Kom}^l(m,r)$ if $k \neq l$ then*

$$B \otimes A \simeq 0.$$

Proof. Observe that composing complexes which factor through projectors P_k and P_l with $k \neq l$ produces a complex containing a turnback; this is contractible see theorem 6.1 section 6. The composite twisted complex lies in the hull of a collection of contractible complexes and therefore it is contractible by lemma 4.9 section 4. □

We will now state what is meant by universal higher order projectors.

Definition 10.4. (universal higher order projectors) A chain complex $P \in \mathrm{Kom}(n)$ is a kth *universal higher order projector* if

(1) The through-degree $\tau(P)$ of P is equal to k.
(2) P vanishes when the number of turnbacks is sufficiently high. For any $l \in \mathbb{Z}_+$ and $a \in \mathrm{Cob}(n,l)$ if $\tau(a) < k$ then

$$a \otimes P \simeq 0 \quad \text{and} \quad P \otimes \bar{a} \simeq 0.$$

(3) There exists a a chain complex $C \in \mathrm{Kom}(n)$ with $\tau(C) < k$ and a twisted complex

$$D = 1_n \to C \to tP$$

such that

$$a \otimes D \simeq 0 \quad \text{and} \quad D \otimes \bar{a} \simeq 0$$

for all diagrams $a \in \mathrm{Cob}(n,m)$ such that $\tau(a) \leq k$.

For each sequence $\epsilon \in \mathcal{L}_n$, there is a complex Q_ϵ (see definition 7.6 section 7.4) and if $|\epsilon| = k$ then Q_ϵ factors through P_k by construction. It follows that any object $A \in \langle Q_\epsilon \rangle$ must factor through P_k. In particular, $P_\epsilon \in \mathrm{Kom}^{|\epsilon|}(n)$ for each $\epsilon \in \mathcal{L}_n$.

As in definition 2.14 section 2.8, the constitutents of the higher order projector $P_{n,k}$ consist of projectors P_ϵ with $\epsilon \in \mathcal{L}_{n,k}$. The categorified construction differs in that there are now non-trivial maps between the components, P_ϵ. We extract $P_{n,k}$ from the resolution of identity in the definition below.

Definition 10.5. ($P_{n,k}$) A kth *universal higher order projector* $P_{n,k}$ is the convolution of the subcomplex formed by isotypic components in the resolution of identity.

$$P_{n,k} = \left(\bigoplus_{\epsilon \in \mathcal{L}_{n,k}} P_\epsilon, d_{\epsilon\nu} \right)$$

Theorem 10.6. *The chain complexes $P_{n,k}$ of definition 10.5 are universal higher order projectors.*

Before proving the theorem we record several useful observations. By construction $P_{n,k}$ is contained in the hull of the set

$$\mathcal{Q} = \{Q_\epsilon : \epsilon \in \mathcal{L}_{n,k}\}.$$

The next observation follows from the discussion preceding the definition.

Observation 5. The kth universal higher order projector factors through the universal projector P_k.

$$P_{n,k} \in \mathrm{Kom}^k(n)$$

Since each $P_{n,k}$ is the restriction of the resolution of identity to the subcomplex consisting of the P_ϵ with $|\epsilon| = k$ we can write a resolution of identity purely in terms of the higher order projectors.

Observation 6.

$$1_n \simeq P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,n-4} \to P_{n,n-2} \to P_{n,n}$$

The diagram on the right usually contains higher differentials, $P_{n,i} \to P_{n,j}$, when $i < j$.

We are ready to prove that the chain complexes $P_{n,k}$ extracted from the resolution of identity above satisfy the properties listed in definition 10.4.

Proof. (of theorem 10.6) The first property, $\tau(P_{n,k}) = k$, follows from observation 5 above.

Suppose that $a \in \mathrm{Cob}(n,l)$ is a diagram with $\tau(a) < k$. Again, by observation 5 the complex $a \otimes P_{n,k}$ is contained in the hull of $\{a \otimes Q_\epsilon : |\epsilon| = k\}$, but $a \otimes Q_\epsilon \simeq 0$ because Q_ϵ factors through P_k. Any complex in the hull of a collection of contractible complexes is contractible by lemma 4.9 section 4.

Rotating distinguished triangles in observation 6 above gives the homotopy equivalence

$$1_n \to t(P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,k-2}) \to P_{n,k} \simeq P_{n,k+2} \to P_{n,n}$$

Let D be the left-hand side of this equation and set C to be the middle term

$$C = t(P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,k-2})$$

so that the third property follows. □

We have seen that the chain complexes $P_{n,k}$ defined above are kth universal higher order projectors. The next theorem states that any chain complex P satisfying the properties of definition 10.4 is homotopy equivalent to the chain complex $P_{n,k}$.

Theorem 10.7. *If $P \in \mathrm{Kom}(n)$ is a kth universal higher order projector then P is homotopy equivalent to $P_{n,k}$ of definition 10.4.*

Proof. Suppose that $P \in \mathrm{Kom}(n)$ satisfies properties (1)-(3) of definition 10.4 above. From observation 6 we have the resolution of identity.

$$1_n \simeq P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,n-4} \to P_{n,n-2} \to P_{n,n}$$

Applying $P \otimes -$ above yields

$$P = P \otimes 1_n \simeq P_{n,k} \otimes P.$$

By property (3) there are complexes C and D where

$$D = 1_n \to C \to tP.$$

Since $\tau(P_{n,k}) = k$, property (3) also implies that $P_{n,k} \otimes D \simeq 0$. Now $P_{n,k} \otimes D \simeq 0$ tells us that

$$0 \simeq P_{n,k} \otimes 1_n \to P_{n,k} \otimes C \to tP_{n,k} \otimes P = \mathrm{Cone}(P_{n,k} \to P_{n,k} \otimes P)$$

because $P_{n,k} \otimes C \simeq 0$. However since $\mathrm{Cone}(f) \simeq 0 \Leftrightarrow f$ is a homotopy equivalence, the above equation implies that $P_{n,k} \otimes P \simeq P$ and therefore $P_{n,k} \simeq P$. □

Now that existence and uniqueness have been shown, we continue our discussion with a series of useful observations.

Proposition 10.8. *The top projector $P_{n,n}$ is a universal projector P_n.*

Proof. This can be seen indirectly by comparing the three properties found in definition 10.4 with those of theorem 6.1 section 6. Alternatively, this can be seen directly by tracing through the construction in either section 8 or section 9. See [6] for an extended discussion of the axioms found in theorem 6.1. □

The universal projectors $P_{n,n}$ of definition 6.1 section 6 were first categorified in [6, 18, 9]. The bottom projectors $P_{2n,0} \in \mathrm{Kom}(n)$ were categorified and related to the Hochschild homology of Khovanov's ring H_n in [19].

Observation 7. The $P_{n,k}$ are mutually orthogonal idempotents,

$$P_{n,k} \otimes P_{n,l} \simeq \delta_{k,l} P_{n,k}.$$

When $k \neq l$ the statement $P_{n,k} \otimes P_{n,l} \simeq 0$ follows from observation 5 and lemma 10.3 above. If $k = l$ then consider the resolution of identity

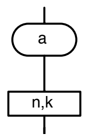
$$1_n \simeq P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,n-2} \to P_{n,n}$$

composing with $P_{n,k}$ gives $P_{n,k} = P_{n,k} \otimes 1_n \simeq P_{n,k} \otimes P_{n,k}$.

Observation 8. Suppose that $a \in \mathrm{Kom}(m,n)$ then

$$a \otimes P_{m,k} \simeq P_{n,k} \otimes a.$$

In pictures,



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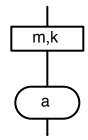

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The proof is analogous to the proof of observation 2 section 2.8 using the resolution of identities in $\mathrm{Kom}(n)$ and $\mathrm{Kom}(m)$.

Theorem 7.9 section 7 implies that Hom-spaces between convolutions of Q_ϵ and Q_ν are contractible when $\epsilon \ntrianglelefteq \nu$. Since the complexes P_ϵ are convolutions of Q_ϵ and $P_{n,k}$ is constructed using P_ϵ with $\epsilon \in \mathcal{L}_{n,k}$,

$$\mathrm{Hom}^*(P_{n,i}, P_{n,j}) \simeq 0 \quad \text{when} \quad j < i.$$

In particular, we have all of the ingredients necessary for an application of theorem 4.10 section 4.

Theorem 10.9. *For each $n > 0$, if $\mathcal{P}^n = \{P_{n,n}, P_{n,n-2}, P_{n,n-4}, \ldots\}$ is the set of higher order projectors then there exists a differential graded algebra,*

$$E^n = \bigoplus_{i \le j} \mathrm{Hom}^*(P_{n,i}, P_{n,j}),$$

and the category of left E^n modules is equivalent to the hull,

$$E^n\text{-mod} \simeq \langle \mathcal{P}^n \rangle.$$

If H_n is Khovanov's ring H_n [16] then the same theorem can be seen to apply with $\mathrm{Kom}^*(n)$ replaced by the dg category of chain complexes of $H_n - H_n$ bimodules.

Using the correspondence,

$$X \in \mathrm{Kom}^*(n) \mapsto X \otimes - \in \mathrm{End}(\mathrm{Kom}^*(n))$$

we may view $\langle \mathcal{P}^n \rangle \subset \mathrm{End}(\mathrm{Kom}^*(n))$ as a subcategory of idempotent functors on $\mathrm{Kom}^*(n)$.

One can use the properties of $P_{n,k}$ discussed above to construct a model for each $P_{n,k}$ as a convolution of chain complexes involving only the universal projector P_k. This implies that the dga $\mathrm{End}^*(P_{n,k})$ is a kind of extension of $\mathrm{End}^*(P_k)$. The precise nature of the algebra $\mathrm{End}^*(P_k)$ as been the focus of a series of conjectures by the authors Gorsky, Oblomkov, Rasmussen and Shende, see [10].

10.10. **Postnikov Decomposition.** In this section we discuss a more homotopy theoretic characterization of the projectors $P_{n,k}$ and the resolution of identity R_n.

From observation 6 above we have

$$1_n \simeq R_n = P_{n,n\,(\mathrm{mod}\,2)} \to \cdots \to P_{n,n-4} \to P_{n,n-2} \to P_{n,n}$$

Truncating the resolution of identity at k yields triangles of inhomogeneous idempotents.

$$R_n = W_{n,k} \to Z_{n,k}$$

where

$$W_{n,k} = P_{n,n\,(\mathrm{mod}\,n)} \to \cdots \to P_{n,k-2} \quad \text{and} \quad Z_{n,k} = P_{n,k} \to P_{n,k+2} \to \cdots \to P_{n,n}.$$

For each k, there is a canonical map, $Z_{n,k} \to Z_{n,k-2}$, which yields a triangle

$$P_{n,k-2} \to Z_{n,k} \to Z_{n,k-2}$$

For any object $S \in \mathrm{Ho}(\mathrm{Kom}(n))$ these triangles can be sewn together in order to obtain the canonical decomposition pictured below.

$$\begin{tikzcd} & S \otimes P_{n,k-2} \arrow[d] & S \otimes P_{n,k-4} \arrow[d] & & \\ \cdots \arrow[r] & S \otimes Z_k \arrow[r] & S \otimes Z_{k-2} \arrow[r] & \cdots \arrow[r] & S \otimes R_n \cong S \end{tikzcd}$$

The picture above is reminiscent of a Postnikov type decomposition. It is well known that such decompositions can be constructed functorially using Bousfield localization. We now define a comparable construction of this decomposition which is functorial in the same sense.

We must extend the category $\mathrm{Kom}(n)$ a bit so that certain limits are guaranteed to exist.

Definition 10.11. ($\mathrm{Kom}^{>\oplus}(n,m)$) Let $\mathrm{Kom}^{>\oplus}(n,m) = \mathrm{Kom}(\mathrm{Cob}(n,m))$ be the category of chain complexes which are bounded in sufficiently negative homological degree and which is closed under small coproducts. We will continue to use the shorthand, $\mathrm{Kom}^{>\oplus}(n) = \mathrm{Kom}^{>\oplus}(n,n)$.

The resolution of identity R_n and the projectors $P_{n,k}$ exist in $\mathrm{Kom}^{>\oplus}(n)$ and the second property of definition 10.4 above implies that functor

$$- \otimes Z_{n,k} : \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n)) \to \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$$

annihilates the full subcategory $\mathrm{Kom}^{>\oplus,<k}(n)$ consisting of chain complexes of diagrams with through-degree less than k.

On the other hand, there is a Bousfield localization,

$$i_{n,k} \circ \pi_{n,k} : \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n)) \to \mathrm{Ho}(\mathrm{Kom}(n))/\,\mathrm{Ho}(\mathrm{Kom}^{>\oplus,<k}(n)) \to \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$$

where $\mathrm{Ho}(\mathrm{Kom}^{>\oplus,<k}(n)) \subset \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$ is the thick subcategory associated to $\mathrm{Kom}^{>\oplus,<k}(n)$, see [17]. Bousfield localization is characterized by the property that triangulated functors

$$F : \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n)) \to \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$$

which annihilate the subcategory $\mathrm{Ho}(\mathrm{Kom}^{>\oplus,<k}(n))$ factor through the localization. In particular, the functor $-\otimes Z_{n,k}$ factors through Bousfield localization.

Alternatively, if $F : \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n)) \to \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$ is a triangulated functor which annihilates $\mathrm{Ho}(\mathrm{Kom}^{>\oplus,<k}(n))$ then writing

$$S = S \otimes 1_n \cong S \otimes R_n \cong S \otimes W_{n,k} \to S \otimes Z_{n,k}$$

for all $S \in \mathrm{Ho}(\mathrm{Kom}^{>\oplus}(n))$ implies that

$$F(S) \cong F(S \otimes W_{n,k} \to S \otimes Z_{n,k}) \cong F(S \otimes Z_{n,k})$$

because $W_{n,k}$ is supported on the subcategory annihilated by F. It follows that F factors through $\mathrm{image}(-\otimes Z_{n,k})$. This suggests the following conjecture.

Conjecture. *The kth Bousfield localization above is equivalent to the kth layer of the Postnikov decomposition associated to the resolution of identity,*

$$i_{n,k} \circ \pi_{n,k} \cong - \otimes Z_{n,k}$$

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